APPENDIX A: DETECTION AND TRACKING ALGORITHM

The formulation of a set of *sufficient* conditions able to discriminate eddies unambiguously, is still a matter of debate in literature. Recent efforts have been dedicated to the construction of condition-free algorithms based on deep learning methods (Ashkezari et al., 2016; Duo et al., 2019; Moschos et al., 2020). Such systems ingest a training set of ”images” of good vortices and try to detect eddies by using classification techniques based mainly on convolutional neural networks. On the other hand, classical detection algorithms, based on the recognition of physical footprints left by eddies, are still largely developed and used in realistic applications, being, so far, more ”mature” and able to reach a satisfactory degree of accuracy. The method used here, belongs to such second group, and consists of a multi-component detection that looks for simultaneous physical/geometrical signatures of the same eddy over different variables (Cipollone et al., 2017). The first component is a well-established geometric identification of local depression/elevation of sea surface height (SSH) that must be dominated by rotation or by streamlines with circular/spiral patterns. Starting from local SSH extremes (minimum and maximum), eddies emerge by cutting subsequent SSH contours towards zero, adding perimetrical area at each step (Chelton et al., 2011b; Xu et al., 2011; Faghmous et al., 2015). A different approach is described in Dong et al. (2012) and Lin et al. (2015), where the eddy perimeter is defined as the outermost enclosed streamlines in the velocity field. The second component looks at the water column underneath and scans the occurrence and persistence of a coherent rotation (same sign, same center) along the vertical layers. There must exist at least one point, say $P(x, y)$, whose surrounding points satisfy clockwise or counterclockwise conditions (clockwise being $u'(x, y+1) > 0 \quad v'(x+1, y) > 0 \quad u'(x, y-1) < 0 \quad v'(x-1, y) < 0$) in the same x, y position from surface to the eddy bottom. If such rotation vanishes within the mixed layer, the corresponding eddy is removed. This choice is imposed to avoid false-positive patterns whose origin is unclear (instabilities in the forcing fields, inconsistencies of model dynamic with assimilation increments, etc.) and to remove eddies that are too shallow to contribute to mixing processes. In Cipollone et al. (2017) it has been shown that about 10% of the surface patterns identified by a pure SSH algorithm does not extend below 5m in a global re-analysis at 1/4°. The third component shapes the vertical structure of the eddy based on the mean rotational velocity v_{R}, defined as :

$$v_{\mathrm{R}} = \hat{z} \cdot \langle \frac{\vec{\boldsymbol{d}} \times \vec{\boldsymbol{v}}}{|\vec{\boldsymbol{d}}|} \delta_{\mathrm{eddy}}(\vec{\boldsymbol{d}}) \rangle_{\mathrm{mean}} \quad \vec{\boldsymbol{d}} = \vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_{\mathrm{m.c.}} \tag{3}$$

where δ_{eddy} is 1 inside the eddy projection from the surface and 0 elsewhere. $\vec{\boldsymbol{r}}_{\mathrm{m.c.}}$ and $\vec{\boldsymbol{d}}$ refer respectively to the eddy mean center and the spatial distance of each point from the center while the spatial sum $\langle ... \rangle$ is therefore restricted to the eddy area. The v_{R} is strictly linked to the angular momentum and it tends to be a conserved quantity as the eddy changes latitude. It does not rely on numerical derivative and it is poorly affected by noise when calculated layer by layer. The v_{R} changes with depth and can be considered a measure of the trapping strength for different layers. It is worth to note that $v_{\mathrm{R}}(\boldsymbol{z})$ is evaluated along the water column using the same horizontal area in a cylindric-like way in order to be consistent with values at different depth. The system relates $v_{\mathrm{R}}(\boldsymbol{z})$ to the fraction of the volume that the eddy is able to confine and drag away. The trapped volume is now mapped by a ”shrinking” distance $\vec{\boldsymbol{d}}(\boldsymbol{z})$:

$$\begin{aligned} \vec{\boldsymbol{d}}(\boldsymbol{z}) &= \vec{\boldsymbol{d}} * f(\boldsymbol{z}) \qquad \text{where} \\ f(\boldsymbol{z}) &= \tfrac{v_{\mathrm{R}}(\boldsymbol{z})}{v_{\mathrm{R}}(\boldsymbol{z}-\mathbf{1})} \qquad \frac{v_{\mathrm{R}}(\boldsymbol{z})}{v_{\mathrm{R}}(\boldsymbol{z}-\mathbf{1})} < 1 \\ f(\boldsymbol{z}) &= 1 \qquad \frac{v_{\mathrm{R}}(\boldsymbol{z})}{v_{\mathrm{R}}(\boldsymbol{z}-\mathbf{1})} > 1 \end{aligned} \tag{4}$$

that is proportional to the relative strength of the rotation with respect to upper layers, i.e. a region of strong angular momentum corresponds to a large fraction of volume trapped. A somewhat similar approach in the velocity field is used in Doglioli et al. (2007) and Lin et al. (2015), where the velocity component must reverse sign passing from one side to the other of the eddy across the eddy center. In Lin et al. (2015), the eddy center is also allowed to drift between two neighboring vertical layers, such drift must not exceed the 25 $\%$ of the eddy radius. Different definitions of trapped volume can be found in literature that consider some fixed predefined functions for the eddy shape or some global/regional thresholds on speed/vorticity. Profiles generated from eq (4) are totally empirical and linked to the strength of $v_{\mathrm{R}}(\boldsymbol{z})$, being therefore very sensible to the specific characteristic of each eddy (cyclonic/anticyclonic, etc.). Fig. 10 shows a schematic representation of the detection system together with an example of the detected population (belonging to June 1^{st}, 2012). Eddies whose areas drop below 1$\%$ within the mixed layer depth are removed. This choice follows the one previously described that removes eddies vanishing within the same depth.

The tracking algorithm concatenates the same eddy in consecutive time records and employs a largely known nearest-neighbour method (Chelton et al., 2007, 2011b), although differing in the following minor technical refinements . Once the eddy centre is found at time t the same eddy is sought at time $t + 1$ in the same position within a box of a radius 100km that roughly corresponds to the mean eddy horizontal extension. Following Chelton et al. (2011b), a second tracking procedure between time-step t and t+2 is implemented to limit missing tracks originating from some eddies standing at step t, disappearing at t + 1 and re-appearing at time step t + 2.

Based on the motility of the eddy in the Mediterranean Sea and the daily frequency of the outputs, we decide to keep fixed the searching area for this second search; other choices in literature consider to expand this research area by 1.5 times (Chelton et al., 2011b; Lin et al., 2015).

This second procedure recovers eddies that tend to appear and disappear frame by frame, due to weak signatures and whose vertical extension oscillates around the mixed layer depth. Moreover it helps to deplete possible noise generated by the adjustment of the assimilation increments added into the model on a weekly basis. Such second procedure leads to about 15% more eddies detected and its impact at global scale has been discussed in Cipollone et al. (2017).

APPENDIX B: RELATIVE EDDY KINETIC ENERGY

Following the approach proposed by Cipollone et al. (2017), who extended the formulation of relative kinetic energy in Chelton et al. (2011b) to take into account three-dimensional structures, we consider the relative eddy kinetic energy (REKE) to represent the fraction of 3D kinetic energy carried by eddies in space and time, i.e. :

$$\begin{aligned}
\text{REKE} &= \text{EKE}_{eddy}/\text{EKE}_{tot} \\
\text{EKE}_{eddy}(x,y) &= \frac{1}{2N}\sum_{i=1}^{N}\int_{\text{depth}}^{0}\delta(x,y,z,t_i)\left[u'(x,y,z,t_i)^2+v'(x,y,z,t_i)^2\right]\mathrm{d}z \\
\text{EKE}_{tot}(x,y) &= \frac{1}{2N}\sum_{i=1}^{N}\int_{\text{depth}}^{0}\left[u'(x,y,z,t_i)^2+v'(x,y,z,t_i)^2\right]\mathrm{d}z
\end{aligned} \tag{5}$$

where N is the number of frames (timesteps), u' and v' are anomalous zonal and meridional velocity (referred to their temporal averages computed over the period 1993-2016), depth is the depth-range considered, and $\delta(x,y,z,t_i)$ is 1 within eddy interior and 0 elsewhere, ρ is the water density. Considering only the top ocean layer, equations in (5) reduce to surface formulation by Chelton et al. (2011b). The definition of the eddy vertical structure follows a bowl-shape structure as function of the rotational velocity as described in the Appendix A.