**Appendix**

It is easy to calculate the characteristic polynomials of the following graphs.

1. The signless Laplacian characteristic polynomials of graphs $T_j^4(n-|T_j^4|,0,\ldots,0)$ for $j=1,2,\ldots,20$.

$$
\begin{aligned}
& Q(T_5^4(n-7,0,\ldots,0))-Q(T_1^4(n-6,0,\ldots,0)) \\
= \;& (x-1)^{n-8}[(2n-8)x^6-(20n-82)x^5+(75n-306)x^4-(131n-502)x^3 \\
& +(107n-330)x^2-(33n-16)x+44], \\
& Q(T_6^4(n-7,0,\ldots,0))-Q(T_1^4(n-6,0,\ldots,0)) \\
= \;& (x-1)^{n-8}[(2n-8)x^6-(20n-82)x^5+(72n-291)x^4-(111n-402)x^3+(66n-137)x^2 \\
& -(9n+60)x+44], \\
& Q(T_{10}^4(n-7,0,\ldots,0))-Q(T_1^4(n-6,0,\ldots,0)) \\
= \;& (x-1)^{n-8}[(n-3)x^6-(10n-32)x^5+(36n-115)x^4-(57n-160)x^3+(39n-62)x^2 \\
& -(9n+24)x+12], \\
& Q(T_{15}^4(n-8,0,\ldots,0))-Q(T_1^4(n-6,0,\ldots,0)) \\
= \;& (x-1)^{n-9}[(3n-13)x^7-(35n-157)x^6+(156n-711)x^5-(341n-1495)x^4+ \\
& (368n-1420)x^3-(176n-396)x^2+(24n+128)x-32] \\
& Q(T_{16}^4(n-8,0,\ldots,0))-Q(T_1^4(n-6,0,\ldots,0)) \\
= \;& (x-1)^{n-9}[(3n-13)x^7-(35n-157)x^6+(156n-711)x^5-(341n-1495)x^4+ \\
& (368n-1420)x^3-(176n-396)x^2+(24n+128)x-32], \\
& Q(T_{17}^4(n-8,0,\ldots,0))-Q(T_1^4(n-6,0,\ldots,0)) \\
= \;& (x-1)^{n-9}[(3n-13)x^7-(35n-157)x^6+(156n-705)x^5-(335n-1459)x^4+ \\
& (361n-1382)x^3-(186n-468)x^2+(36n+64)x-48], \\
& Q(T_7^4(n-8,0,\ldots,0))-Q(T_4^4(n-7,0,\ldots,0)) \\
= \;& (x-1)^{n-9}[(2n-10)x^7-(24n-122)x^6+(110n-561)x^5-(239n-1197)x^4+ \\
& (246n-1160)x^3-(102n-404)x^2+(12n-32)x], \\
& Q(T_8^4(n-8,0,\ldots,0))-Q(T_4^4(n-7,0,\ldots,0)) \\
= \;& (x-1)^{n-9}[(2n-10)x^7-(24n-122)x^6+(113n-579)x^5-(265n-1353)x^4+ \\
& (324n-1622)x^3-(194n-928)x^2+(44n-192)x], \\
& Q(T_9^4(n-8,0,\ldots,0))-Q(T_4^4(n-7,0,\ldots,0)) \\
= \;& (x-1)^{n-9}[(n-4)x^7-(12n-50)x^6+(55n-233)x^5- \\
& (121n-505)x^4+(132n-518)x^3-(66n-224)x^2+(12n-32)x], \\
& Q(T_{18}^4(n-9,0,\ldots,0))-Q(T_4^4(n-7,0,\ldots,0)) \\
= \;& (x-1)^{n-10}[(4n-23)x^8-(53n-308)x^7+(283n-1655)x^6-(787n-4609)x^5+ \\
& (1226n-7150)x^4-(1063n-6137)x^3+(474n-2710)x^2-(88n-520)x], \\
& Q(T_{19}^4(n-9,0,\ldots,0))-Q(T_4^4(n-7,0,\ldots,0)) \\
= \;& (x-1)^{n-10}[(2n-9)x^8-(27n-126)x^7+(146n-698)x^6-(406n-1962)x^5+ \\
& (620n-2980)x^4-(512n-3202)x^3+(208n-1722)x^2-(32n-128)x], \\
& Q(T_{20}^4(n-9,0,\ldots,0))-Q(T_4^4(n-7,0,\ldots,0)) \\
= \;& (x-1)^{n-10}[(3n-16)x^8-(41n-224)x^7+(223n-1235)x^6-(618n-3426)x^5+
\end{aligned}
$$

$$
\begin{aligned}
& (930n-5080)x^4-(752n-3968)x^3+(304n-1520)x^2-(48n-224)x], \\
& Q(T_{12}^4(n-8,0,\ldots,0))-Q(T_{14}^4(n-7,0,\ldots,0)) \\
= \;& (x-1)^{n-9}[(n-4)x^7-(12n-48)x^6+(54n-207)x^5-(115n-387)x^4+ \\
& (120n-277)x^3-(57n-7)x^2+(9n+72)x-12], \\
& Q(T_{13}^4(n-7,0,\ldots,0))-Q(T_{14}^4(n-7,0,\ldots,0)) \\
= \;& (x-1)^{n-8}[(n-5)x^6-(10n-50)x^5+(35n-171)x^4-(50n-222)x^3+(22n-50)x^2-32x].
\end{aligned}
$$

2. The signless Laplacian characteristic polynomials of graphs $T_j^6(n-|T_j^6|,0,\ldots,0)$ for $j=1,2,\ldots,24$

$$
\begin{aligned}
& Q(T_3^6(n-7,0,\ldots,0))-Q(T_1^6(n-5,0,\ldots,0)) \\
= \;& (x-1)^{n-8}[(2n-7)x^6-(21n-78)x^5+(76n-282)x^4-(120n-414)x^3 \\
& +(82n-219)x^2-(19n-4)x+12] \\
& Q(T_4^6(n-5,0,\ldots,0))-Q(T_1^6(n-5,0,\ldots,0)) \\
= \;& (x-1)^{n-6}[x^4-(n+2)x^3+(2n+9)x^2-(n+20)x+12] \\
& Q(T_5^6(n-6,0,\ldots,0))-Q(T_1^6(n-5,0,\ldots,0)) \\
= \;& (x-1)^{n-7}[(n-2)x^5-(10n-26)x^4+(29n-72)x^3-(30n-46)x^2+(9n+20)x-12] \\
& Q(T_6^6(n-6,0,\ldots,0))-Q(T_1^6(n-5,0,\ldots,0)) \\
= \;& (x-1)^{n-7}[(n-2)x^5-(10n-28)x^4+(30n-92)x^3-(34n-104)x^2+(12n-32)x] \\
& Q(T_7^6(n-6,0,\ldots,0))-Q(T_1^6(n-5,0,\ldots,0)) \\
= \;& (x-1)^{n-7}[(n-2)x^5-(10n-28)x^4+(29n-67)x^3-(31n-25)x^2+(10n+56)x-32] \\
& Q(T_8^6(n-7,0,\ldots,0))-Q(T_1^6(n-5,0,\ldots,0)) \\
= \;& (x-1)^{n-8}[(2n-6)x^6-(22n-74)x^5+(83n-281)x^4-(138n-422)x^3+(100n-200)x^2 \\
& -(24n+48)x+32] \\
& Q(T_9^6(n-7,0,\ldots,0))-Q(T_1^6(n-5,0,\ldots,0)) \\
= \;& (x-1)^{n-8}[(2n-6)x^6-(22n-74)x^5+(82n-278)x^4-(132n-408)x^3+(91n-157)x^2 \\
& -(21n+53)x+12] \\
& Q(T_{11}^6(n-7,0,\ldots,0))-Q(T_1^6(n-5,0,\ldots,0)) \\
= \;& (x-1)^{n-8}[(2n-6)x^6-(22n-74)x^5+(82n-281)x^4-(131n-420)x^3+(88n-216)x^2 \\
& -20nx+16] \\
& Q(T_{12}^6(n-7,0,\ldots,0))-Q(T_1^6(n-5,0,\ldots,0)) \\
= \;& (x-1)^{n-8}[(2n-6)x^6-(22n-76)x^5+(84n-309)x^4-(143n-536)x^3+(108n-384)x^2 \\
& -(28n-64)x+16] \\
& Q(T_{13}^6(n-8,0,\ldots,0))-Q(T_1^6(n-5,0,\ldots,0)) \\
= \;& (x-1)^{n-9}[(3n-11)x^7-(37n-145)x^6+(171n-675)x^5-(391n-1463)x^4+(477n-1518)x^3 \\
& -(320n-588)x^2+(80n+112)x-112] \\
& Q(T_{14}^6(n-6,0,\ldots,0))-Q(T_1^6(n-5,0,\ldots,0)) \\
= \;& (x-1)^{n-7}[(2n-7)x^5-(19n-71)x^4+(58n-209)x^3-(71n-209)x^2+(30n-16)x-48] \\
& Q(T_{15}^6(n-7,0,\ldots,0))-Q(T_1^6(n-5,0,\ldots,0)) \\
= \;& (x-1)^{n-8}[(3n-12)x^6-(33n-140)x^5+(129n-557)x^4-(230n-974)x^3+(188n-728)x^2 \\
& -(56n-144)x+32]
\end{aligned}
$$

$$
\begin{aligned}
& Q(T^6_{16}(n-7,0,\ldots,0)) - Q(T^6_1(n-5,0,\ldots,0)) \\
= \; & (x-1)^{n-8}[(3n-12)x^6-(33n-140)x^5+(129n-553)x^4-(223n-948)x^3+(175n-694)x^2 \\
& -(49n-152)x+12] \\
& Q(T^6_{17}(n-7,0,\ldots,0)) - Q(T^6_1(n-5,0,\ldots,0)) \\
= \; & (x-1)^{n-8}[(3n-12)x^6-(33n-138)x^5+(128n-526)x^4-(224n-830)x^3+(178n-461)x^2 \\
& -(51n+68)x+92] \\
& Q(T^6_{18}(n-8,0,\ldots,0)) - Q(T^6_1(n-5,0,\ldots,0)) \\
= \; & (x-1)^{n-9}[(4n-18)x^7-(50n-232)x^6+(238n-1089)x^5-(567n-2411)x^4+(727n-2560)x^3 \\
& -(484n-960)x^2+(132n+304)x-240] \\
& Q(T^6_{19}(n-8,0,\ldots,0)) - Q(T^6_1(n-5,0,\ldots,0)) \\
= \; & (x-1)^{n-9}[(4n-18)x^7-(50n-234)x^6+(238n-1117)x^5-(567n-2561)x^4+(724n-2936)x^3 \\
& -(474n-1444)x^2+(124n-32)x-128] \\
& Q(T^6_{20}(n-8,0,\ldots,0)) - Q(T^6_1(n-5,0,\ldots,0)) \\
= \; & (x-1)^{n-9}[(4n-18)x^7-(50n-234)x^6+(238n-1118)x^5-(566n-2560)x^4+(722n-2931)x^3 \\
& -(477n-1461)x^2+(129n-48)x-140] \\
& Q(T^6_{21}(n-9,0,\ldots,0)) - Q(T^6_1(n-5,0,\ldots,0)) \\
= \; & (x-1)^{n-10}[(5n-25)x^8-(70n-366)x^7+(387n-2077)x^6-(1106n-5990)x^5 \\
& +(1768n-9465)x^4-(1579n-8096)x^3+(728n-3349)x^2-(133n-420)x+44] \\
& Q(T^6_{22}(n-9,0,\ldots,0)) - Q(T^6_1(n-5,0,\ldots,0)) \\
= \; & (x-1)^{n-10}[(5n-25)x^8-(70n-368)x^7+(388n-2119)x^6-(1117n-6302)x^5 \\
& +(1810n-10535)x^4-(1648n-9917)x^3+(776n-4848)x^2-(144n-924)x+16] \\
& Q(T^6_{23}(n-9,0,\ldots,0)) - Q(T^6_1(n-5,0,\ldots,0)) \\
= \; & (x-1)^{n-10}[(5n-25)x^8-(70n-368)x^7+(388n-2083)x^6-(1071n-5888)x^5 \\
& +(1616n-8789)x^4-(1277n-6578)x^3+(461n-2013)x^2-(52n-96)x+16] \\
& Q(T^6_{24}(n-10,0,\ldots,0)) - Q(T^6_1(n-5,0,\ldots,0)) \\
= \; & (x-1)^{n-11}[(6n-29)x^9-(97n-489)x^8+(635n-3295)x^7-(2193n-11569)x^6 \\
& +(4340n-22917)x^5-(4963n-25577)x^4+(3101n-14800)x^3-(898n-3378)x^2 \\
& +(72n+16)x-16]
\end{aligned}
$$