Supplementary Material

1 DEFINITIONS AND NOTATIONS

1.1 Fractional Derivatives

1.1.1 The Fourier transform and the Riesz-Feller space-fractional derivative

Let (S1) be the Fourier transform of a general function $f(x)$,

$$\widehat{f}(\kappa) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{+\infty} \mathrm{e}^{i\kappa x} f(x)dx, \quad \kappa \in \mathbb{R} \tag{S1}$$

and let (S2),

$$f(x) = \mathcal{F}^{-1}\{\widehat{f}(\kappa)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-i\kappa x}\widehat{f}(\kappa)d\kappa, \quad x \in \mathbb{R} \tag{S2}$$

be the inverse Fourier transform. For a sufficiently well-behaved function $f(x)$ we define the Riesz-Feller space-fractional derivative of order α and skewness θ as

$$\begin{cases} \mathcal{F}\{{}_x\mathcal{D}_\theta^\alpha f(x);\kappa\} = \psi_\alpha^\theta(\kappa)\widehat{f}(\kappa), \quad \psi_\alpha^\theta(\kappa) = -|\kappa|^\alpha \mathrm{e}^{i(\mathrm{sign}(\kappa))\theta\pi/2} \\ 0 < \alpha \leqslant 2, \quad |\theta| \leqslant \min\{\alpha, 2-\alpha\} \end{cases} \tag{S3}$$

$$\begin{aligned} {}_x\mathcal{D}_\theta^\alpha f(x) = \frac{\Gamma(1+\alpha)}{\pi}\{\sin[(\alpha+\theta)\pi/2]\int_0^\infty \frac{f(x+\xi)-f(x)}{\xi^{1+\alpha}}d\xi \\ + \sin[(\alpha-\theta)\pi/2]\int_0^\infty \frac{f(x-\xi)-f(x)}{\xi^{1+\alpha}}d\xi\}. \end{aligned} \tag{S4}$$

The symbol $\psi_\alpha^\theta(\kappa)$ is the logarithm of the characteristic function of a general Levy strictly stable probability density with index of stability α and asymmetry parameter θ (improperly called skewness) according to Feller's parameterization.

1.1.2 The Laplace transform and the Caputo fractional derivative

Let

$$\widetilde{f}(s) = \mathcal{L}\{f(t)\} = \int_0^\infty \mathrm{e}^{-st} f(t)dt, \quad \Re(s) > a_f, \tag{S5}$$

be the Laplace transform of a function $f(t)$, and let

$$f(t) = \mathcal{L}^{-1}\{\tilde{f}(s)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \mathrm{e}^{st}\tilde{f}(s)ds, \quad \Re(s) = \gamma > a_f \tag{S6}$$

where $t > 0$ and a_f is a constant defined such that the product $e^{-a_f t}|f(t)|$ is bounded for all t greater than some T (i.e., the constant a_f exists provided the existence of the Laplace transform). For a sufficiently well-behaved function $f(t)$ we define the Caputo time-fractional derivative of order β, $(0 < \beta \leqslant 1)$ through

$$\mathcal{L}\{{}_t\mathcal{D}_*^\beta f(t)\} = s^\beta \widetilde{f}(s) - s^{\beta-1}f(0^+), \quad 0 < \beta \leqslant 1 \tag{S7}$$

Hence, we can write

$$ {}_t\mathcal{D}_*^{\beta} f(t) = \begin{cases} \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{f^{(1)}(\tau)}{(t-\tau)^{\beta}} d\tau, & 0 < \beta < 1 \\ \frac{d}{dt} f(t), & \beta = 1 \end{cases} . \tag{S8} $$

1.2 Stable distribution: Sato et al. (1999); Nolan (2003); Feller (1962)

A non-degenerate random variable X is called stable if for all $n > 1$, there exist constants $c_n > 0$ and $d_n \in \mathbb{R}$ such that $X_1 + \cdots + X_n \stackrel{d}{=} c_n X + d_n$[1], where $X_1, X_2, \cdots, X_n$ are i.i.d realizations of X. The random variable X is strictly stable if and only if $d_n = 0, \forall n$. It can be shown that the only possible choice for the scaling constants is $c_n = n^{1/\alpha}$ for a certain value $\alpha \in (0, 2]$.

1.2.1 Parameterizations of stable laws:

There are different parameterizations for stable distribution. The variety of parameterizations is caused by a combination of historical evolution, plus the numerous problems that have been analyzed applying specialized forms of stable distributions.

1. A random variable X is stable if and only if $X \stackrel{d}{=} aZ + b$ where $a \neq 0, b \in \mathbb{R}$ and Z is a random variable with characteristic function (Nolan (2003))

$$ \mathbb{E}[\exp(i\kappa Z)] = \begin{cases} \exp(-|\kappa|^{\alpha}[1 - i\zeta \tan\frac{\pi\alpha}{2}(\operatorname{sign}\kappa)]) & \alpha \neq 1 \\ \exp(-|\kappa|[1 + i\zeta\frac{2}{\pi}(\operatorname{sign}\kappa)\log|\kappa|]) & \alpha = 1 \end{cases} , \tag{S9} $$

   where $0 < \alpha \leqslant 2$, $-1 \leqslant \zeta \leqslant 1$[2].

2. A random variable X is parameterized as $S(\alpha, \zeta, c, \mu; 1)$ if

$$ \mathbb{E}(\exp(i\kappa X)) = \exp(i\kappa\mu - |c\kappa|^{\alpha}(1 - i\zeta \operatorname{sign}(\kappa)\Phi_1)), \tag{S10} $$

   where,

$$ \Phi_1 = \begin{cases} \tan(\frac{\pi\alpha}{2}) & \alpha \neq 1 \\ -\frac{2}{\pi}\log|\kappa| & \alpha = 1 \end{cases} . \tag{S11} $$

   The distribution is assumed to be standard when the scale $c = 1$ and the location $\mu = 0$ (Nolan (2003)).

3. A random variable X is $S(\alpha, \zeta, \gamma, \delta; 0)$ (Nolan (2003)) if

$$ X \stackrel{d}{=} \begin{cases} \gamma\left(Z - \beta\tan\frac{\pi\alpha}{2}\right) + \delta & \alpha \neq 1 \\ \gamma Z + \delta & \alpha = 1 \end{cases} , \tag{S12} $$

   where the Z is defined at (S9). This can also be rewritten as:

$$ \mathbb{E}(\exp(i\kappa X)) = \exp(i\kappa\delta - |\gamma\kappa|^{\alpha}(1 - i\zeta \operatorname{sign}(\kappa)\Phi_0)), \tag{S13} $$

   where

$$ \Phi_0 = \begin{cases} (1 - |\gamma\kappa|^{1-\alpha})\tan(\frac{\pi\alpha}{2}) & \alpha \neq 1 \\ -\frac{2}{\pi}\log|\gamma\kappa| & \alpha = 1 \end{cases} . \tag{S14} $$

   This form ($S(\alpha, \zeta, \gamma, \delta; 0)$) is continuous at $\alpha = 0$. Note that this form is the one used in MATLAB.

[1] The symbol $\stackrel{d}{=}$ designates the equality in distribution.

[2] The parameter ζ is usually called β but β is used to describe another parameter in this manuscript.

4. Feller's parameterization: (Sato et al. (1999); Feller (1962); Gorenflo and Mainardi (1999); Takayasu (1990); Mainardi et al. (2007)) A random variable X is stable if and only if $X \stackrel{d}{=} aY + b$ where $0 < \alpha \leqslant 2$, $\theta \leqslant \min(\alpha, 2-\alpha)$, $a \neq 0$, $b \in \mathbb{R}$ and Y is a random variable with characteristic function

$$\mathbb{E}(\exp(i\kappa Y)) = \exp(i\psi_\alpha^\theta(\kappa)), \tag{S15}$$

where $\psi_\alpha^\theta(\kappa)$ is given by (S3). It is also worth to mention that for $b = 0$ the characteristic function of X (which is strictly stable) is given as

$$\mathbb{E}(\exp(i\kappa X)) = \exp\left(i\psi_\alpha^\theta\left(\frac{\kappa}{a}\right)\right) = \exp\left(i|a|^\alpha\psi_\alpha^\theta(\kappa)\right). \tag{S16}$$

For the sake of performing simulation using existing methods on MATLAB, we have to express the Feller's parameterization in the $S(\alpha, \zeta, \gamma, \delta; 0)$ form. First, we are interested in the strictly stable case ($\delta = \zeta\gamma\tan(\frac{\pi\alpha}{2})$) so we have

$$\exp(i|a|^\alpha\psi_\alpha^\theta(\kappa)) = \exp\left(i\kappa\zeta\gamma\tan\left(\frac{\pi\alpha}{2}\right) - |\gamma\kappa|^\alpha(1 - i\zeta\,\mathrm{sign}(\kappa)\Phi)\right). \tag{S17}$$

The above equation should be correct for any $\kappa \in \mathbb{R}$. Solving them (considering separate equation for imaginary and real parts) gives

$$\begin{aligned}
\gamma &= a\left(\cos\left(\frac{\pi\theta}{2}\right)\right)^{1/\alpha} \\
\zeta &= -\tan\left(\frac{\pi\theta}{2}\right)\cot\left(\frac{\pi\alpha}{2}\right) \\
\delta &= \zeta\gamma\tan\left(\frac{\pi\alpha}{2}\right) = -a\tan\left(\frac{\pi\theta}{2}\right)\left(\cos\left(\frac{\pi\theta}{2}\right)\right)^{1/\alpha}.
\end{aligned} \tag{S18}$$

1.2.2 Fractional order absolute moment

Suppose the characteristic function of random variable X is denoted as $\varphi_X(\kappa) = \mathbb{E}[\exp(i\kappa X)]$. Applying the method described in (Harvill (2009)) and using its general result (S20), the fractional order absolute moment of stable distributions are computed.
Define an auxiliary function $\rho(.)$:

$$\rho(\delta) = \int_0^\infty u^{-(\delta+1)}\sin^2(u)\mathrm{d}u = \begin{cases} \delta^{-1}2^{\delta-1}\Gamma(1-\delta)\cos(\pi\delta/2), & \text{if } 0 < \delta < 2, \delta \neq 1 \\ \pi/2, & \text{if } \delta = 1 \end{cases}. \tag{S19}$$

The general result of (Harvill (2009)) is:

$$\rho(\delta)\mathbb{E}[|X|^\delta] = -\frac{1}{4}\int_0^\infty \kappa^{-(\delta+1)}[\varphi_X(2\kappa) + \varphi_X(-2\kappa) - 2]\mathrm{d}\kappa. \tag{S20}$$

When X is a strictly stable random variable with decomposing the characteristic function as

$$\varphi_X(\kappa) = \exp\{-z_p\kappa^\alpha\} \quad \text{and} \quad \varphi_X(-\kappa) = \exp\{-z_n\kappa^\alpha\}, \quad \kappa \geqslant 0 \tag{S21}$$

where,

$$z_p = \exp(i\theta\pi/2), \quad z_n = \exp(-i\theta\pi/2). \tag{S22}$$

Then

$$\rho(\delta)\mathbb{E}[|X|^\delta] = \frac{1}{\delta}2^{\delta-2}\Gamma\left(1-\frac{\delta}{\alpha}\right)\left(z_p^{\delta/\alpha} + z_n^{\delta/\alpha}\right). \tag{S23}$$

Therefore, the absolute moment of order δ is given as

$$\mathbb{E}[|X|^\delta] = \frac{\Gamma\left(1-\frac{\delta}{\alpha}\right)\cos\left(\frac{\delta\pi\theta}{2\alpha}\right)}{\Gamma(1-\delta)\cos\left(\frac{\delta\pi}{2}\right)}. \tag{S24}$$

1.2.3 Signed fractional order moment

Using the method provided in Kuruoglu (2001), the signed absolute moment or order δ for α-stable distribution is written as follows. For $\delta \in (-2,-1) \cup (-1,0)$ we have

$$\begin{aligned}
\mathbb{E}[X^{\langle\delta\rangle}] &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \operatorname{sign}(x)|x|^\delta \varphi_X(\kappa)e^{i\kappa x}dx d\kappa \\
&= \frac{i}{\pi}\int_0^{\infty}\int_0^{\infty} x^\delta \sin(\kappa x)dx\left(\varphi_X(\kappa) - \varphi_X^*(-\kappa)\right)dt \qquad \text{(S25)} \\
&= \frac{i}{\pi}\Gamma(1+\delta)\cos\left(\frac{\delta\pi}{2}\right)\int_0^{\infty}\kappa^{-1-\delta}\left(e^{-\kappa^\alpha z_p} - e^{-\kappa^\alpha z_n}\right)dt \\
&= \frac{i}{\pi}\Gamma(1+\delta)\cos\left(\frac{\delta\pi}{2}\right)\left[\frac{1}{\alpha}\Gamma\left(-\frac{\delta}{\alpha}\right)\left(z_p^{\frac{\delta}{\alpha}} - z_n^{\frac{\delta}{\alpha}}\right)\right] \\
&= -\frac{\Gamma\left(1-\frac{\delta}{\alpha}\right)\sin\left(\frac{\delta\pi\theta}{2\alpha}\right)}{\Gamma(1-\delta)\sin\left(\frac{\delta\pi}{2}\right)}. \qquad \text{(S26)}
\end{aligned}$$

As discussed in Kuruoglu (2001), the same result can be generalized to $\delta \in [0,\alpha]$.

1.3 Lévy stable stochastic processes: Zolotarev (1986); Sato et al. (1999)

A one-dimensional stochastic process $\{X(t); t \geqslant 0\}$ said to be a Lévy process if it satisfies the following properties:

1. $X(0) \stackrel{a.s.}{=} 0$.
2. Disjoint increments are mutually independent. It means that for any $0 \leqslant t_1 < t_2 < \cdots < t_n < \infty$ the increments $(X(t_2) - X(t_1), X(t_3) - X(t_2), \ldots, X(t_n) - X(t_{n-1}))$ are mutually independent.
3. Stationary increments: for any $s < t$, $X(t) - X(s)$ is equal in distribution to $X(t-s)$.
4. The sample paths are Cádlág (Billingsley (2008)), meaning they are almost surely right-continuous and have left limits at all time points.

A process $X(t)$ is said to be a $strictly\,\alpha - stable\,process$ if it is a Lévy process which also satisfies the scaling (self-similarity) property (i.e., the process $(c\,X(t\,c^{-\alpha}); t \geqslant 0)$ has the same distribution as $X(t)$ for every $c > 0$ denoted as $X(t) \overset{d}{=} t^{1/\alpha} X(1)$ (Kyprianou (2006)).

One of the main property of a Lévy process is that its characteristic function has the following form

$$\mathbb{E}[\exp(i\,\kappa\,X(t))] = \exp(t\,\Psi(\kappa)). \tag{S27}$$

This means that the stationary independent increments of process $X(t)$ are i.i.d samples of a stable distribution. In case of a one dimensional $strictly\,\alpha - stable\,process$ with asymmetric parameter θ, the $\Psi(\kappa)$ is equal to $\psi_{\theta}^{\alpha}(\kappa)$.

1.3.1 Space fractional diffusion: (Gorenflo and Mainardi (2012); Mainardi et al. (2007); Leonenko et al. (2014))

One group of random processes that have a space fractional diffusion are the strictly α-stable processes. Suppose the random process $X(t_*)$ is a strictly α-stable process for some $0 < \alpha \leqslant 2$ and $0 \leqslant |\theta| \leqslant \min(\alpha, 2-\alpha)$. Then, according to (S27), the diffusion that is defined as $f_{\alpha,\theta}(x, t_*) = \mathcal{P}\{X(t_*) = x|X(0) = 0\}$, has a Fourier transform equal to

$$\widehat{f}_{\alpha,\theta}(\kappa, t_*) = \exp(-t_*\,\psi_{\alpha}^{\theta}(\kappa)), \tag{S28}$$

where $\psi_{\alpha}^{\theta}(\kappa)$ is the same function defined at (S3). Taking the Laplace transform on t_*, the fractional order PDE of the diffusion is achieved

$$\widetilde{\widehat{f}}_{\alpha,\theta}(\kappa, s_*) = \frac{1}{s_* + \psi_{\alpha}^{\theta}(\kappa)}, \tag{S29}$$

and then

$$-\,\psi_{\alpha}^{\theta}(\kappa)\,\widetilde{\widehat{f}}_{\alpha,\theta}(\kappa, s_*) = s_*\,\widetilde{\widehat{f}}_{\alpha,\theta}(\kappa, s_*) - 1. \tag{S30}$$

So

$$\begin{aligned} \frac{\partial}{\partial t_*} f_{\alpha,\theta}(x, t_*) &= {}_x\mathcal{D}_{\theta}^{\alpha}\{f_{\alpha,\theta}(x, t_*)\}, \quad t_* \geqslant 0 \\ f_{\alpha,\theta}(x, 0) &= \delta(x), \quad x \in \mathbb{R}. \end{aligned} \tag{S31}$$

1.3.2 Stable subordinator process: Meerschaert and Straka (2013); Leonenko et al. (2014)

A Subordinator process is defined as a Lévy process with non-decreasing sample paths. Suppose $T(t_*)$ is strictly β-stable[3] process for some $0 < \beta \leqslant 1$, and $\theta = -\beta$. It can be shown that the condition $\theta = -\beta$ (which is only feasible when $0 < \beta \leqslant 1$) implies that the increments of the process $T(t_*)$ are almost surely non-negative, thus here we use the Laplace transform. The diffusion defined as $r_{\beta}(t, t_*) = \mathcal{P}\{T(t_*) = t|T(0) = 0\}$, has a Laplace transform equal to

$$\widetilde{r}_{\beta}(s, t_*) = \exp(-t_* s^{\beta}). \tag{S32}$$

[3] An α-stable process where $\alpha = \beta$

1.3.3 Inverse subordinator: Meerschaert and Straka (2013); Gorenflo and Mainardi (2012)

Because $T(t_*)$ is a monotonically increasing function, the inverse process $(T_*(t))$ is a well defined function, which could be interpreted as the first hitting time.

$$T_*(t) = \inf\{\tau \quad |T(\tau) \geqslant t\}, \tag{S33}$$

which can be used to write the following properties.

$$\begin{gathered} t_2 > t_1 \Longrightarrow T_*(t_2) \geqslant T_*(t_1), \\ \mathcal{P}(T_*(t) \leqslant t_*) = \mathcal{P}(T(t_*) \geqslant t). \end{gathered} \tag{S34}$$

Define $q_\beta(t_*, t) = \mathcal{P}\{T_*(t) = t_* | T_*(0) = 0\}$. Using the property in (S34):

$$\int_0^{t_*} q_\beta(t'_*, t) dt'_* = \int_t^\infty r_\beta(t', t_*) dt'. \tag{S35}$$

So

$$q_\beta(t_*, t) = \frac{\partial}{\partial t_*} \int_t^\infty r_\beta(t', t_*) dt' = \int_t^\infty \frac{\partial}{\partial t_*} r_\beta(t', t_*) dt'. \tag{S36}$$

Then

$$\widetilde{q}_\beta(t_*, s) = \frac{-1}{s} \frac{\partial}{\partial t_*} \widetilde{r}_\beta(s, t_*) = s^{\beta-1} \exp(-t_* s^\beta). \tag{S37}$$

1.4 Space-Time fractional diffusion: Gorenflo and Mainardi (2012), Mainardi et al. (2007), Saichev and Zaslavsky (1997),Gorenflo et al. (2000), Scalas et al. (2000), Metzler and Klafter (2000)

Suppose $X(t_*)$ is a strictly α-stable process and $T(t_*)$ is a subordinator process. The process $X(t) = X(T_*(t))$ is called a subordinated Leonenko et al. (2014) process if $T_*(t)$ be the inverse process of the subordinator process $(T(t_*))$ [4]. Define a diffusion function $u(x, t) = \mathcal{P}\{X(t) = x\}$, hence a direct result of the definition is

$$u(x, t) = \int_0^\infty f_{\alpha,\theta}(x, t_*) q_\beta(t_*, t) dt_*, \tag{S38}$$

where $f_{\alpha,\theta}(x, t_*)$ and $q_\beta(t_*, t)$ defined in previous section. Using the Laplace and Fourier transform, the aforementioned expression can be simplified

$$\begin{aligned} \widehat{\widetilde{u}}(\kappa, s) &= \int_0^\infty \widehat{f}_{\alpha,\theta}(\kappa, t_*) \widetilde{q}_\beta(t_*, s) dt_* \\ &= \int_0^\infty [\exp(-\psi_\alpha^\theta(\kappa))][s^{\beta-1} \exp(-t_* s^\beta)] dt_* \\ &= \frac{s^{\beta-1}}{s^\beta + \psi_\alpha^\theta(\kappa)}. \end{aligned} \tag{S39}$$

[4] t_* is named operational time while t is called physical/regular time

Then,

$$s^{\beta}\widehat{\widetilde{u}}(\kappa,s)-s^{\beta-1}=-\psi_{\alpha}^{\theta}(\kappa)\widehat{\widetilde{u}}(\kappa,s). \tag{S40}$$

Thus,

$${}_{t}\mathcal{D}_{*}^{\beta}u(x,t)={}_{x}\mathcal{D}_{\theta}^{\alpha}u(x,t),\quad u(x,0)=\delta(x),\quad x\in\mathbb{R},\quad t\geqslant 0, \tag{S41}$$

where $0<\alpha\leqslant 2, |\theta|\leqslant\min\{\alpha,2-\alpha\}$ and $0<\beta\leqslant 1$. One important result of (S39) is the scaling property of the diffusion function which can be reflected by a single variable function $K_{\alpha,\beta}^{\theta}(x)$

$$u(x,t)=t^{-\gamma}u(x/t^{\gamma},1)=t^{-\gamma}K_{\alpha,\beta}^{\theta}(x/t^{\gamma}),\quad \gamma=\beta/\alpha. \tag{S42}$$

The following properties of $K_{\alpha,\beta}^{\theta}(x)$ will be used later (Mainardi et al. (2007))

$$K_{\alpha,\beta}^{\theta}(-x)=K_{\alpha,\beta}^{-\theta}(x) \tag{S43}$$

$$\begin{cases}\int_0^{+\infty}K_{\alpha,\beta}^{\theta}(x)x^{\delta}\mathrm{d}x=\rho\frac{\Gamma(1-\delta/\alpha)\Gamma(1+\delta/\alpha)\Gamma(1+\delta)}{\Gamma(1-\rho\delta)\Gamma(1+\rho\delta)\Gamma(1+\beta\delta/\alpha)}\\ \\ -\min\{\alpha,1\}<\Re(\delta)<\alpha,\quad \rho=\frac{\alpha-\theta}{2\alpha}.\end{cases} \tag{S44}$$

2 PROOFS OF THE PROPOSITION

2.1 Proof of Proposition 1

PROOF. Using equation (S38), we write that

$$\begin{aligned}\mathbb{E}[|X(t)|^{\delta}] &= \int_{-\infty}^{\infty}|x|^{\delta}u(x,t)dx\\ &= \int_{-\infty}^{\infty}\int_{0}^{\infty}|x|^{\delta}f_{\alpha,\theta}(x,t_*)q_{\beta}(t_*,t)dt_*dx.\end{aligned} \tag{S45}$$

Now, using the discussion in Section 1.2.2, and using $\varphi_{X(t_*)}(\kappa)=\exp\left(t_*\psi_{\alpha}^{\theta}(\kappa)\right)$ from (S27) for $D=1$, and $\varphi_{X(t_*)}(\kappa)=\exp\left(t_*D\psi_{\alpha}^{\theta}(\kappa)\right)$ for $D\neq 1$. After substituting in (S21), we write that

$$\int_{\infty}^{\infty}|x|^{\delta}f_{\alpha,\theta}(x,t_*)dx=t_*^{\frac{\delta}{\alpha}}D^{\frac{\delta}{\alpha}}\frac{\Gamma\left(1-\frac{\delta}{\alpha}\right)\cos\left(\frac{\delta\pi\theta}{2\alpha}\right)}{\Gamma(1-\delta)\cos\left(\frac{\delta\pi}{2}\right)}. \tag{S46}$$

Using (S46) and (S37), we continue with the Laplace transform of the time-varying moment expression in (S45 as

$$\begin{aligned}\mathbb{E}[\widetilde{|X(t)|^{\delta}}] &= \int_{-\infty}^{\infty}\int_{0}^{\infty}|x|^{\delta}f_{\alpha,\theta}(x,t_*)q_{\beta}(t_*,s)dt_*dx\\ &= \int_{0}^{\infty}t_*^{\frac{\delta}{\alpha}}D^{\frac{\delta}{\alpha}}\frac{\Gamma(1-\frac{\delta}{\alpha})\cos\left(\frac{\delta\pi\theta}{2\alpha}\right)}{\Gamma(1-\delta)\cos\left(\frac{\delta\pi}{2}\right)}s^{\beta-1}\exp\left(-t_*s^{\beta}\right)dt_*\end{aligned}$$

$$= \quad s^{-(\delta\frac{\beta}{\alpha}+1)} D^{\frac{\delta}{\alpha}} \frac{\Gamma\left(1-\frac{\delta}{\alpha}\right)\Gamma\left(1+\frac{\delta}{\alpha}\right)\cos\left(\frac{\delta\pi\theta}{2\alpha}\right)}{\Gamma(1-\delta)\cos\left(\frac{\delta\pi}{2}\right)}. \tag{S47}$$

Using the inverse Laplace relation $\mathcal{L}^{-1}\{s^{-\left(\delta\frac{\beta}{\alpha}+1\right)}\} = \frac{t^{\delta\frac{\beta}{\alpha}}}{\Gamma\left(1+\delta\frac{\beta}{\alpha}\right)}$, and taking inverse Laplace transform on both sides in (S47), we finally write the time-varying moment with order δ as

$$\mathbb{E}[|X(t)|^{\delta}] = t^{\delta\frac{\beta}{\alpha}} D^{\frac{\delta}{\alpha}} \frac{\Gamma\left(1-\frac{\delta}{\alpha}\right)\Gamma\left(1+\frac{\delta}{\alpha}\right)\cos\left(\frac{\delta\pi\theta}{2\alpha}\right)}{\Gamma(1-\delta)\Gamma\left(1+\delta\frac{\beta}{\alpha}\right)\cos\left(\frac{\delta\pi}{2}\right)}. \tag{S48}$$

■

2.2 Proof of Proposition 2

PROOF. Using equation (S38), we write that

$$\begin{aligned}
\mathbb{E}[X(t)^{\langle\delta\rangle}] &= \int_{-\infty}^{\infty} x^{\langle\delta\rangle} u(x,t)dx \\
&= \int_{-\infty}^{\infty}\int_{0}^{\infty} \text{sign}(x)|x|^{\delta} f_{\alpha,\theta}(x,t_*) q_{\beta}(t_*,t)dt_* dx.
\end{aligned} \tag{S49}$$

Now, using the discussion in Section 1.2.3, and using $\varphi_{X(t_*)}(\kappa) = \exp\left(t_*\psi_{\alpha}^{\theta}(\kappa)\right)$ from (S27) for $D=1$, and $\varphi_{X(t_*)}(\kappa) = \exp\left(t_* D\psi_{\alpha}^{\theta}(\kappa)\right)$ for $D \neq 1$. After substituting in (S25), we write that

$$\int_{\infty}^{\infty} \text{sign}(x)|x|^{\delta} f_{\alpha,\theta}(x,t_*)dx = -t_*^{\frac{\delta}{\alpha}} D^{\frac{\delta}{\alpha}} \frac{\Gamma\left(1-\frac{\delta}{\alpha}\right)\sin\left(\frac{\delta\pi\theta}{2\alpha}\right)}{\Gamma(1-\delta)\sin\left(\frac{\delta\pi}{2}\right)}. \tag{S50}$$

Using (S50) and (S37), we continue with the Laplace transform of the time-varying signed moment expression in (S49 as

$$\begin{aligned}
\mathbb{E}[\widetilde{X(t)^{\langle\delta\rangle}}] &= \int_{-\infty}^{\infty}\int_{0}^{\infty} x^{\langle\delta\rangle} f_{\alpha,\theta}(x,t_*) q_{\beta}(t_*,s)dt_* dx \\
&= -\int_{0}^{\infty} t_*^{\frac{\delta}{\alpha}} D^{\frac{\delta}{\alpha}} \frac{\Gamma\left(1-\frac{\delta}{\alpha}\right)\sin\left(\frac{\delta\pi\theta}{2\alpha}\right)}{\Gamma(1-\delta)\sin\left(\frac{\delta\pi}{2}\right)} s^{\beta-1}\exp(-t_* s^{\beta})dt_* \\
&= -s^{-\left(\delta\frac{\beta}{\alpha}+1\right)} D^{\frac{\delta}{\alpha}} \frac{\Gamma\left(1-\frac{\delta}{\alpha}\right)\Gamma\left(1+\frac{\delta}{\alpha}\right)\sin\left(\frac{\delta\pi\theta}{2\alpha}\right)}{\Gamma(1-\delta)\sin\left(\frac{\delta\pi}{2}\right)}.
\end{aligned} \tag{S51}$$

Using the inverse Laplace relation $\mathcal{L}^{-1}\{s^{-(\delta\frac{\beta}{\alpha}+1)}\} = \frac{t^{\delta\frac{\beta}{\alpha}}}{\Gamma(1+\delta\frac{\beta}{\alpha})}$, and taking inverse Laplace transform on both sides in (S51), we finally write the time-varying signed moment with order δ as

$$\mathbb{E}[X(t)^{\langle\delta\rangle}] = -t^{\delta\frac{\beta}{\alpha}} D^{\frac{\delta}{\alpha}} \frac{\Gamma(1-\frac{\delta}{\alpha})\Gamma(1+\frac{\delta}{\alpha})\sin(\frac{\delta\pi\theta}{2\alpha})}{\Gamma(1-\delta)\Gamma(1+\delta\frac{\beta}{\alpha})\sin(\frac{\delta\pi}{2})}. \tag{S52}$$

■

2.3 Proof of Proposition 3

PROOF. Using equation (S38), we write that

$$\begin{aligned}\mathbb{E}[\log|X(t)|] &= \int_{-\infty}^{\infty} \log|x| u(x,t) dx \\ &= \int_{-\infty}^{\infty}\int_{0}^{\infty} \log|x| f_{\alpha,\theta}(x,t_*) q_\beta(t_*,t) dt_* dx. \end{aligned} \tag{S53}$$

For writing the log moments, we observe that $\mathbb{E}[\exp(t\log|X|)] = \mathbb{E}[|X|^t]$, i.e., the moment generating function of $\log|X|$ is the absolute moment of order t. Since we know the expression for absolute moments of alpha stable distributions from Section 1.2.2, we use the following to write the log moments

$$\mathbb{E}[(\log|X|)^n] = \lim_{\delta\to 0}\frac{d^n}{d\delta^n}\mathbb{E}[|X|^\delta].$$

Using the similar procedure as mentioned in Kuruoglu (2001), we can write that

$$\mathbb{E}[|X|^\delta] = c(\delta) K^\delta \Gamma\left(1-\frac{\delta}{\alpha}\right)\cos(\delta R), \tag{S54}$$

where,

$$c(\delta) = \Gamma(1+\delta)\mathrm{sinc}\left(\frac{\pi\delta}{2}\right), \qquad K = (t_* D)^{\frac{\delta}{\alpha}}, \qquad R = \frac{\pi\theta}{2\alpha}. \tag{S55}$$

We then write that

$$\frac{d}{d\delta}\mathbb{E}[|X(t_*)|^\delta] = h(\delta)\mathbb{E}[|X(t_*)|^\delta], \tag{S56}$$

where,

$$h(\delta) = \frac{1}{\alpha}\log(t_* D) + \frac{c'(\delta)}{c(\delta)} - \frac{1}{\alpha}\psi^{(0)}\left(1-\frac{\delta}{\alpha}\right) - R\tan(\delta R),$$

and we have used the polygamma function as

$$\psi^{(n-1)}(x) = \frac{d^n}{dx^n}\log(\Gamma(x))$$

Therefore, the expected log moment is written as

$$\mathbb{E}[\log|X(t_*)|] = h(0) = \frac{1}{\alpha}\log(t_* D) + \psi^{(0)}(1)\left(1 - \frac{1}{\alpha}\right). \tag{S57}$$

Using (S57) and (S37), we continue with the Laplace transform of the time-varying log moment expression in (S53) as

$$\begin{aligned}\mathbb{E}[\widetilde{\log|X(t)|}] &= \int_{-\infty}^{\infty}\int_{0}^{\infty} \log|x| f_{\alpha,\theta}(x, t_*) q_\beta(t_*, s) dt_* dx \\ &= \int_{0}^{\infty}\left(\frac{1}{\alpha}\log(t_* D) + \psi^{(0)}(1)\left(1 - \frac{1}{\alpha}\right)\right) s^{\beta-1}\exp\left(-t_* s^\beta\right) dt_*. \end{aligned}\tag{S58}$$

Now, using the Euler-Mascheroni constant and the following integral expression

$$\int_{0}^{\infty} \log(x) e^{-x} dx = -\gamma,$$

we can write the Laplace of the log moment as

$$\mathbb{E}[\widetilde{\log|X(t)|}] = \left(\frac{\log(D)}{\alpha} + \psi^{(0)}(1)(1 - \frac{1}{\alpha}) - \frac{\gamma}{\alpha}\right)\frac{1}{s} - \frac{\beta}{\alpha s}\log(s). \tag{S59}$$

Inverting the Laplace transform in (S59) using the identity $\mathcal{L}^{-1}\left\{\frac{\log(s)}{s}\right\} = -\gamma - \log(t)$, and using the fact that $\psi^{(0)}(1) = -\gamma$, we have

$$\mathbb{E}[\log|X(t)|] = \frac{\beta}{\alpha}\log(t) + \frac{\log(D)}{\alpha} + \gamma\left(\frac{\beta}{\alpha} - 1\right). \tag{S60}$$

■

2.4 Proof of Proposition 4

PROOF. Using the similar approach as in proof of the Proposition 3 and as derived in Kuruoglu (2001), we can write that

$$\mathrm{var}(\log|X(t_*)|) = \psi^{(1)}(1)\left(\frac{1}{2} + \frac{1}{\alpha}^2\right) - \left(\frac{\pi\theta}{2\alpha}\right)^2. \tag{S61}$$

Using the expression in (S61), we can write the Laplace of the variance of log absolute values as

$$\begin{aligned}\mathrm{var}(\widetilde{\log|X(t)|}) &= \int_{-\infty}^{\infty} \mathrm{var}(\log|X(t)|) q_\beta(t_*, s) dt_* \\ &= \int_{-\infty}^{\infty}\left(\psi^{(1)}(1)\left(\frac{1}{2} + \frac{1}{\alpha}^2\right) - \left(\frac{\pi\theta}{2\alpha}\right)^2\right) s^{\beta-1}\exp\left(-t_* s^\beta\right) dt_*\end{aligned}$$

$$\stackrel{(a)}{=} \quad \frac{\pi^2}{6}\left(\frac{1}{2}+\frac{1}{\alpha}^2\right)-\left(\frac{\pi\theta}{2\alpha}\right)^2, \tag{S62}$$

where in (a) we substitute the value of polygamma function $\psi^{(1)}(1) = \pi^2/6$. ■

2.5 Proof of Proposition 5

PROOF. Using equation (S38), we write that

$$\begin{aligned} \mathbb{E}[(\log|X(t)|)^2] &= \int_{-\infty}^{\infty} (\log|x|)^2 u(x,t)dx \\ &= \int_{-\infty}^{\infty}\int_{0}^{\infty} (\log|x|)^2 f_{\alpha,\theta}(x,t_*) q_\beta(t_*,t) dt_* dx. \end{aligned} \tag{S63}$$

Using the similar approach as in the proof of the Proposition 3, we can write the second moment of the log absolute values of $X(t_*)$, using (S57) and (S61), as the following

$$\begin{aligned} \mathbb{E}[(\log|X(t_*)|)^2] &= (\mathbb{E}[\log|X(t_*)|])^2 + \text{var}(\log|X(t_*)|) \\ &= \frac{1}{\alpha^2}(\log(t_*))^2 + \frac{2c}{\alpha}\log(t_*) + k, \end{aligned} \tag{S64}$$

where, $c = \frac{\log(D)}{\alpha} + \gamma(\frac{\beta}{\alpha}-1)$ and $k = c^2 + \frac{\pi^2}{6}(\frac{1}{2}+\frac{1}{\alpha}^2) - (\frac{\pi\theta}{2\alpha})^2$. Using the expression in (S64), we can write the Laplace of the second moment of the log absolute values as the following

$$\begin{aligned} \mathbb{E}[\widetilde{(\log|X(t)|)^2}] &= \int_{-\infty}^{\infty}\int_{0}^{\infty} (\log|x|)^2 f_{\alpha,\theta}(x,t_*) q_\beta(t_*,s) dt_* dx \\ &= \int_{0}^{\infty} \left(\frac{1}{\alpha^2}(\log(t_*))^2 + \frac{2c}{\alpha}\log(t_*) + k\right) s^{\beta-1}\exp\left(-t_* s^\beta\right) dt_*. \end{aligned} \tag{S65}$$

Now, using the Euler-Mascheroni constant and the following integral expressions

$$\int_0^\infty \log(x)e^{-x}dx = -\gamma, \qquad \int_0^\infty \log^2(x)e^{-x}dx = \gamma^2 + \frac{\pi^2}{6},$$

we can write the Laplace of the second log moment as

$$\begin{aligned} \mathbb{E}[\widetilde{(\log|X(t)|)^2}] &= \left(k - \frac{2c}{\gamma} + \frac{1}{\alpha^2}\left(\gamma^2+\frac{\pi^2}{6}\right)\right)\frac{1}{s} + \left(\frac{2\beta\gamma}{\alpha^2} - \frac{2\beta c}{\alpha}\right)\frac{\log(s)}{s} \\ &\quad + \frac{\beta^2}{\alpha^2}\frac{\log^2(s)}{s}. \end{aligned} \tag{S66}$$

Inverting the Laplace transform in (S66) using the identity $\mathcal{L}^{-1}\left\{\frac{\log(s)}{s}\right\} = -\gamma - \log(t)$, $\mathcal{L}\left\{\log^2(t)\right\} = \frac{1}{s}\left(\gamma^2 + \frac{\pi^2}{6}\right) + 2\gamma\frac{\log(s)}{s} + \frac{\log^2(s)}{s}$, we have

$$\mathbb{E}[(\log|X(t)|)^2] = \frac{\beta^2}{\alpha^2}\log^2(t) + 2\frac{\beta\gamma}{\alpha}\left(\frac{\beta}{\alpha} - 1\right)\log(t) + c_1, \tag{S67}$$

where $c_1 = \frac{\pi^2}{6}\left(\frac{1}{\alpha^2} + \frac{1}{2}\right) - \left(\frac{\pi\theta}{2\alpha}\right)^2 + \left(\frac{\log(D)}{\alpha} + \gamma\left(\frac{\beta}{\alpha} - 1\right)\right)^2 + \frac{\pi^2}{6\alpha^2}(1-\beta^2)$. ■

3 NUMERICAL SIMULATIONS: DATA GENERATION

A Lévy process can be seen as the limit of a continuous random walk where the number of jumps goes to infinite so both $X(t_*)$ and $T(t_*)$ can be simulated by a continuous random walk for a large number of jumps (Gorenflo and Mainardi (2012)). Define $t_{n*} = n \cdot \Delta_*$ and Δ_* is a constant. Define $T_N = T(t_{n*}) = T(n \times \Delta_*)$ and $X_n = X(t_{n*}) = X(n \times \Delta_*)$. $X_n - X_{n-1}$ and $T_n - T_{n-1}$ are increments of their Lévy processes so they are independent (condition 2), also, they have same distribution (condition 3 3). By adding the condition 1 we have

$$X_n = \sum_i^n \chi_i; \quad n \geqslant 0, \tag{S68}$$

χ are i.i.d samples of an α-stable distribution with skewness parameter θ and scale coefficient $c = (\Delta_*)^{(1/\alpha)}$.

$$T_n = \sum_i^n \tau_i; \quad n \geqslant 0, \tag{S69}$$

τ are i.i.d samples of an β-stable distribution with skewness parameter $-\beta$ and scale coefficient $c = (\Delta_*)^{(1/\beta)}$. Finally, we have

$$X(t) = X_{N(t)}; \quad N(t) = \min\{n:\ t \leqslant T_n\}. \tag{S70}$$

The MATLAB's `random` function, which support stable distribution internally, is used to generate i.i.d samples of a stable distribution. Notice that equations (S18) has been used to change the parameterization.

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