Appendix: the derivation of spectral amplification factor and signal-to-noise ratio

According to the linear approximation theory, the output spike train of the Eq. (5) in the time domain has the form

$$y_i(t) = y_i^0(t) + R(\bar{\mu}, D, t) * (f(t) - \langle f(t) \rangle_0 + s(t)) \tag{A1}$$

where $y_i^0(t)$ represents a realization of the spike train generated by the integrate-and-fire neuron obeying Eq. (A1) in the absence of the time-dependent perturbation, $*$ denotes the convolution, $\bar{\mu} = \mu + \langle f(t) \rangle_0$ is the base current, $R(\bar{\mu}, D, t)$ is the base current and noise level-dependent linear response function, with Fourier transform called linear susceptibility to be explicitly given below, and $\langle f(t) \rangle_0 = Gr(\bar{\mu}, D)$ is the stationary average of the feedback term with the stationary firing rate $r(\bar{\mu}, D)$, which can be found by solving the self-consistent relation [20,27,44]

$$r(\bar{\mu}, D, V_T) = r_0(\bar{\mu} + Gr(\bar{\mu}, D), D, V_T) \tag{A2}$$

with $r_0(\mu, D, V_T) = (\tau_R + \sqrt{\pi}\int_{(\mu - v_T)/\sqrt{2D}}^{(\mu - v_R)/\sqrt{2D}} dz e^{z^2} erfc(z))^{-1}$. Introducing Fourier transform of the zero average spike trains as

$$\tilde{y}_i(\omega) = \frac{1}{\sqrt{T}} \int_0^T dt e^{i\omega t} (y_i(t) - r(\bar{\mu}, D)) .$$

Then, from Eq. (A1), one can get the linear response in the frequency domain as follows:

$$\tilde{y}_i(\omega) = \tilde{y}_i^0(\omega) + A(\omega, \bar{\mu}, D, V_T)[\tilde{s}(\omega) + \frac{G}{N} F(\omega) \sum_{j=1}^{N} \tilde{y}_j(\omega)] \tag{A3}$$

where $\tilde{y}_i^0(\omega) = \mathscr{F}[y_i^0(t)]$ represents a realization of the spiking output of the *i*th neuron in the frequency domain, $F(\omega) = e^{i\omega\tau_D} / (1 - i\omega\tau_S)^2$ is the Fourier transform of the kernel in Eq. (2), and correspondingly $\frac{1}{N} F(\omega) \sum_{j=1}^{N} \tilde{y}_j(\omega)$ stands for the Fourier transform of the zero static mean synaptic feedback $f(t) - \langle f(t) \rangle_0$, and $A(\omega, \mu, D) = \mathscr{F}[R(\bar{\mu}, D, t)]$ is the linear susceptibility given by [27]

$$A(\omega, \mu, D, V_T) = \frac{r(\mu, D) i\omega}{\sqrt{D}(i\omega - 1)} \frac{\tilde{D}_{i\omega - 1}(\frac{\mu - v_T}{\sqrt{D}}) - e^{\gamma} \tilde{D}_{i\omega - 1}(\frac{\mu - v_R}{\sqrt{D}})}{\tilde{D}_{i\omega}(\frac{\mu - v_T}{\sqrt{D}}) - e^{\gamma} e^{i\omega\tau_R} \tilde{D}_{i\omega}(\frac{\mu - v_R}{\sqrt{D}})} \tag{A4}$$

with $\gamma = [v_R^2 - v_T^2 + 2\mu(v_T - v_R)]/4D$. Let

$$Y(\omega) = [\tilde{y}_1(\omega), \tilde{y}_2(\omega), \ldots, \tilde{y}_N(\omega)]^T, \; Y_0(\omega) = [\tilde{y}_1^0(\omega), \tilde{y}_2^0(\omega), \ldots, \tilde{y}_{\mathrm{N}}^0(\omega)]^T .$$

Rewrite Eq. (A3) as

$$Y(\omega) = Y_0(\omega) + A(\omega,\bar{\mu},D,V_T)\tilde{s}(\omega)\alpha + \frac{1}{N}GA(\omega,\bar{\mu},D,V_T)F(\omega)\alpha\alpha^T Y(\omega) \tag{A5}$$

Then, by $\left\langle Y_0(\omega)\tilde{s}^*(\omega)\alpha \right\rangle = 0$,

$$\begin{aligned}\left\langle Y(\omega)Y^H(\omega)\right\rangle &= \left(I + \frac{1}{N}\frac{GA(\omega,\bar{\mu},D,V_T)F(\omega)\alpha\alpha^T}{1-GA(\omega,\bar{\mu},D,V_T)F(\omega)} \right)\left(\left\langle Y_0(\omega)Y_0^H(\omega)\right\rangle + \left|A(\omega,\bar{\mu},D,V_T)\right|^2 \left\langle \tilde{s}(\omega)\tilde{s}^*(\omega)\right\rangle \alpha\alpha^T \right) \\ &\times \left(I + \frac{1}{N}\frac{GA^*(\omega,\bar{\mu},D,V_T)F^*(\omega)\alpha\alpha^T}{1-GA^*(\omega,\bar{\mu},D,V_T)F^*(\omega)} \right)\end{aligned}$$

where the superscripts H and $*$ denote transpose conjugate and conjugate, respectively, and $\alpha = [1,1,\ldots,1]^T$ is a constant auxiliary vector.

Note that for a homogeneous network, the population activity $y(t) = \frac{1}{N}\sum y_i(t)$ is of central importance. From the definition of spectral density $G_{yy}(\omega) = \frac{1}{N^2}\alpha^T \left\langle Y(\omega)Y^H(\omega)\right\rangle \alpha$, one can obtain

$$\begin{aligned} G_{yy}(\omega) &= \left(1+\frac{GA(\omega,\bar{\mu},D,V_T)F(\omega)}{1-GA(\omega,\bar{\mu},D,V_T)F(\omega)}\right)\left(\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left\langle y_i^0(\omega)y_i^{0*}(\omega)\right\rangle + \left|A(\omega,\bar{\mu},D,V_T)\right|^2\left\langle \tilde{s}(\omega)\tilde{s}^*(\omega)\right\rangle\right)\left(1+\frac{GA^*(\omega,\bar{\mu},D,V_T)F^*(\omega)}{1-GA^*(\omega,\bar{\mu},D,V_T)F^*(\omega)}\right) \\ &= \frac{\left|A(\omega,\bar{\mu},D,V_T)\right|^2\left\langle \tilde{s}(\omega)\tilde{s}^*(\omega)\right\rangle + \frac{1}{N}S_0((\omega,\bar{\mu},D,V_T))}{\left|1-GA(\omega,\bar{\mu},D,V_T)F(\omega)\right|^2}. \end{aligned} \tag{A6}$$

Let $S_1(\omega) = \frac{|A|^2}{|1-GAF|^2}\left\langle \tilde{s}\tilde{s}^*\right\rangle$ be the spectral density of the signal component, and $S_2(\omega) = \frac{1}{N|1-GAF|^2}S_0(\omega,\bar{\mu},D,V_T)$ with

$$S_0(\omega,\mu,D,V_T) \overset{\Delta}{=} \left\langle \tilde{y}_i^0(\omega)\tilde{y}_i^{0*}(\omega)\right\rangle = r(\mu,D)\frac{|\tilde{D}_{i\omega}(\frac{\mu-v_T}{\sqrt{D}})|^2 - e^{2\gamma}|\tilde{D}_{i\omega}(\frac{\mu-v_R}{\sqrt{D}})|^2}{|\tilde{D}_{i\omega}(\frac{\mu-v_T}{\sqrt{D}}) - e^{\gamma}e^{i\omega t_R}\tilde{D}_{i\omega}(\frac{\mu-v_R}{\sqrt{D}})|^2} \tag{A7}$$

representing spectral density of fluctuations, then $G_{yy}(\omega) = S_1(\omega) + S_2(\omega)$. With the spectral density $G_{yy}(\omega)$ available, the spectral amplification factor and signal-to-noise ratio can be obtained accordingly, as shown in Eqs.(6) and (7).