Appendix: Langevin dynamics driven by a telegraphic active noise

Jaegon Um,[1,*] Taegun Song,[1,†] and Jae-Hyung Jeon[1,‡]

[1] *Department of Physics, POSTECH, Pohang, South Korea*

A1. DERIVATION OF THE PROPAGATOR EQ. (7)

Here we present the detail of derivation for the propagator Eq. (7) from Eq. (6) through the gaussian integrals. For this purpose, we use an useful identity relation in the gaussian integral

$$\mathcal{N}\mathcal{N}_1 \int dX_1\, e^{-(X_2+aX_1+b)^2/c} e^{-(X_1+A_1X_0+B_1)^2/C_1} = \mathcal{N}_2 e^{-(X_2+A_2X_0+B_2)^2/C_2}\,, \tag{A1}$$

where the normalization constants are $\mathcal{N} = 1/\sqrt{c\pi}$, $\mathcal{N}_1 = 1/\sqrt{C_1\pi}$, and $\mathcal{N}_2 = 1/\sqrt{C_2\pi}$, and the relations between A_i, B_i, and C_i [$i = 1$] are given by

$$A_{i+1} = -aA_i\,, \quad B_{i+1} = -aB_i + b\,, \quad C_{i+1} = a^2C_i + c\,. \tag{A2}$$

We apply for this integral rule to X_2 in Eq. (A1) such that the R.H.S term is integrated over X_2 with a weight function $\mathcal{N}e^{-(X_3+aX_2+b)^2/c}$. Then, we get $\mathcal{N}_3 e^{-(X_3+A_3X_0+B_3)^2/C_3}$ with the recursion relation Eq. (A2) [$i = 2$] and $\mathcal{N}_3 = 1/\sqrt{C_3\pi}$.

First, we consider the propagator $\Pi[v(t_i + \tau_i)|v(t_i)]$ between two successive transition times where $f(t) = f_i$ during the time interval $t \in [t_i, t_i + \tau_i]$, see Fig. 1. Putting $A_1 = a = \gamma\delta t - 1$, $B_1 = b = -f_i\delta t$, and $C_1 = c = 4\gamma\beta^{-1}\delta t$, and repeating the integrations over $X_\mu = v(t_i + \mu\delta t)$, where $\mu \in \{0, \ldots, n\}$ is the index used for discretizing the time interval $\tau_i = n\delta t$ with $n \to \infty$, one can obtain $\Pi\left[v(t_i + \tau_i)|v(t_i)\right] = \sqrt{1/[C_n\pi]}e^{-[v(t_i+\tau_i)+A_nv(t_i)+B_n]^2/C_n}$ with

$$\begin{aligned} A_n &= -(-a)^n = -e^{-\gamma\tau_i}\,, \\ B_n &= b\,\frac{1-(-a)^n}{1+a} = -f_i\left(1 - e^{-\gamma\tau_i}\right)/\gamma\,, \\ C_n &= c\,\frac{1-a^{2n}}{1+a^2} = 2\beta^{-1}\left(1 - e^{-2\gamma\tau_i}\right)\,. \end{aligned} \tag{A3}$$

It is straightforward to obtain the propagator $\Pi\left[v(t+\Delta t)|v(t)\right]$ for an arbitrary time interval Δt from the above relation. For all the transition times $t_i, t_{i+1}, \ldots, t_j$ in between t and $t + \Delta t$, we repeat the same gaussian integral to get $\Pi\left[v(t+\Delta t)|v(t)\right] = \int dv(t_j)\cdots\int dv(t_i)\Pi[v(t + \Delta t)|v(t_j)]\Pi[v(t_j)|v(t_{j-1})]\cdots\Pi[v(t_{i+1})|v(t_i)]\Pi[v(t_i)|v(t)]$. After this integration, we arrive at the expression shown in Eq. (7).

A2. GENERATION OF THE NOISE $f(t)$

The telegraphic noise $f(t)$ in our model is generated by realizing the two random sequences of the noise amplitude $\{f_i\}$ and duration time $\{\tau_i\}$. The random number f_i is i.i.d. from the PDF $\mathcal{P}(f)$ and τ_i is i.i.d. from the PDF of $P(\tau)$.

A. The noise duration time $\{\tau_i\}$

The sequence $\{\tau_i\}$ is generated for the three distinct PDFs of $P(\tau)$ as summarized in Tab. 1. The exponentially distributed random number was generated using the uniform random number in $[0, 1)$ via the inverse transform sampling [1]. The gaussian random number was generated using the standard Box-Müller transform [2]. For the power-law case, we considered the PDF $P(\tau) = \frac{\alpha}{\tau_{\min}(1-(\tau_{\min}/T)^\alpha)}(\tau/\tau_{\min})^{-(1+\alpha)}$ for $\tau \in [\tau_{\min}, T]$ and $\tau_{\min} = 1$. The random number governed by this $P(\tau)$ was generated through its inverse cumulative function $\{X|X = \tau_{\min}u^{-1/\alpha}, X \le T\}$ where u is a uniformly distributed random number $u \in [0, 1)$.

B. The noise amplitude $\{f_i\}$

We considered three distinct PDFs of $\mathcal{P}(f)$ for generating the random sequence $\{f_i\}$. Our primary model for the multiple-states f_i is a uniform distribution in an interval ($f_i \in [-f_0, f_0]$). The uniformly distributed random number was generated using the method in Numerical Recipes in Fortran [2]. The second model is a gaussian model for the distribution of $\{f_i\}$, implemented using the same Box-Müller method above [2]. The third one is the dichotomous noise model $\mathcal{P}(f) = \frac{1}{2}\delta(f + f_0) + \frac{1}{2}\delta(f - f_0)$, which was implemented for the study of two-state systems including the noisy Lévy walks (Figs. 7 and 8).

In Fig. 2, we present the simulated trajectories of $f(t)$ for the three PDFs of $P(\tau)$ [poissonian, gaussian, and power-law] for the choice of the uniform distribution $\mathcal{P}(f)$.

C. Ergodicity relation

For a dynamic quantity Q of interest its ensemble-averaged quantity $\langle Q\rangle$ over the active noise $f(t)$ is obtained by taking the average over both the amplitude $\{f_i\}$ and duration time $\{\tau_i\}$, namely, $\langle Q\rangle = \langle Q\rangle_{f_i,\tau_i}$. For the $f(t)$ considered in our study, we test whether the ergodic relation holds such that the average over $\{\tau_i\}$ is replaced by the time average, that is, $\langle Q\rangle_{f_i,\tau_i} = \overline{\langle Q\rangle}_{f_i}$. In Sec. 2 the ensemble-averaged autocorrelation of $f(t)$, $\langle f(t)f(t+\Delta t)\rangle_{f_i,\tau_i}$, is evaluated within Eq. (3) which is the time-averaged one $\overline{\langle f(t)f(t+\Delta t)\rangle}_{f_i}$. We here examine the ergodic relation of this quantity. For this, in Fig. A1, we evaluate both quantities for increasing lengths of t and T, and check the convergence of both quantities as t and T are

* slung@postech.ac.kr
† judah1982@gmail.com
‡ jeonjh@postech.ac.kr

increased. In each plot, the dotted lines are the ensemble-averaged autocorrelation $\langle f(t)f(t+\Delta t)\rangle_{f_i,\tau_i}$ at $t = 10,\ 10^3$, and 10^5, which were numerically obtained through the average of Eq. (2) over 10^5 sequences of $\{\tau_i\}$. The solid lines are Eq. (3), $\overline{\langle f(t)f(t+\Delta t)\rangle_{f_i}}$ for $T = 10,\ 10^3$, and 10^5. The three panels, respectively, present the cases for the poissonian (Left), gaussian (Middle), and the power-law of $1 < \alpha < 2$ (Right) PDFs that used in our study. The results convincingly show that the ergodicity holds for the $f(t)$ investigated throughout the study; for the poissonian and gaussian $P(\tau)$ both averages are almost indistinguishable at the investigated times, demonstrating that ergodicity is fulfilled as quickly as t (or T) is larger than the average duration time. For the power-law case, the convergence is slower than the two cases; one can see that both quantities are closer to each other as t and T are increasing. When a sufficiently large time is reached ($t = 10^5$) both quantities are converged.

A3. LANGEVIN DYNAMICS SIMULATIONS

Once a set of the active noise $f(t)$ are prepared by the protocols described above, it is straightforward to carry out the numerical simulation of the Langevin equations (1) & (27). For the underdamped Langevin model Eq. (1), we solve it in the scheme of the Euler method and recursively obtain the velocity and position of a particle in unit of timestep $\delta t(= 0.001)$ as in the following:

$$v(t+\delta t) = (1-\delta t)v(t) + [\sqrt{2/\delta t}\zeta + f(t)]\delta t, \qquad \text{(A4)}$$

$$x(t+\delta t) = x(t) + v(t)\delta t, \qquad \text{(A5)}$$

where we put $m = \beta = \gamma = 1$ and ζ is a random number from a gaussian PDF of unit variance. The Box-Müller method was used for generating the gaussian random number [2]. During the integration the evolution of $f(t)$ is given by the chosen sample $f(t)$ prepared in advance. For the overdamped Langevin equation (27), the position of a particle is obtained through the recursive relation

$$x(t+\delta t) = x(t) + [\sqrt{2/\delta t}\zeta + f(t)/\gamma]\delta t \qquad \text{(A6)}$$

where we put $\gamma = 1$ and $D = 1/[\gamma\beta] = 1$.

[1] L. Devroye, *Non-Uniform Random Variate Generation* (Springer-Verlag, New York, NY, USA, 1986).

[2] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes 3rd Edition: The Art of Scientific Computing*, 3rd ed. (Cambridge University Press, New York, NY, USA, 2007).

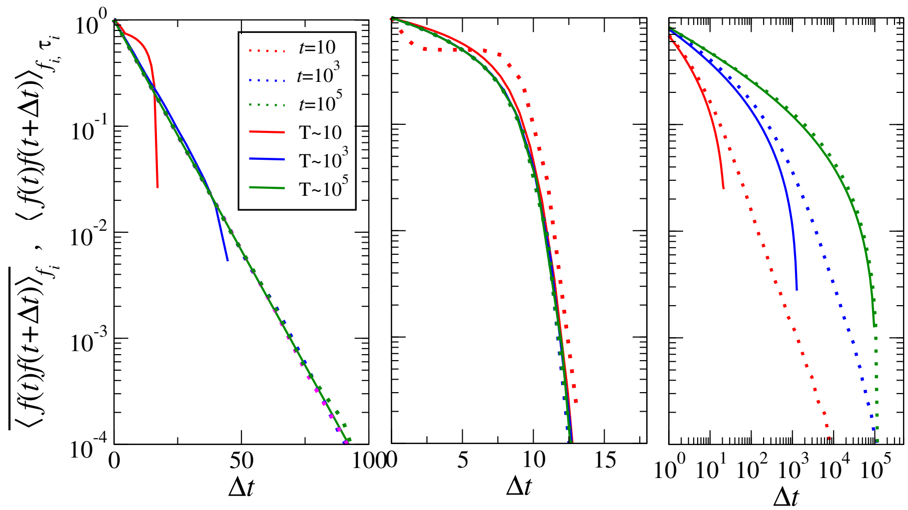

FIG. A1. Ergodicity test for the autocorrelation function of the active noise $f(t)$. The ensemble-averaged autocorrelation $\langle f(t)f(t+\Delta t)\rangle_{f_i,\tau_i}$ is compared to the time-averaged counterpart $\overline{\langle f(t)f(t+\Delta t)\rangle_{f_i}}$ (Eq. (3) from a single $f(t)$ in $[0,\ T]$). From Left to Right, the results correspond to the cases: poissonian (Left), gaussian (Middle), and the power-law (Right). In the plots, the dotted lines indicate the ensemble-averaged autocorrelation at $t = 10$ (blue), 10^3 (magenta), and 10^5 (dark green). It was obtained by averaging Eq. (2) over 10^5 sequences of $\{\tau_i\}$. Solid lines represent the time-averaged autocorrelation, Eq. 2), for $T = 10$ (red), 10^3 (violet) and 10^5 (cyan). The active noises $f(t)$ were generated with the parameters: $\sigma = 1$, $\tau_c = 10$ (poissonian) $\sigma_\tau = 1$ (gaussian), and $\alpha = 1.2$ (power-law).