Appendix A: Appendix: Stream function

A general relation between the velocity field and the stream function ψ can be expressed by

$$\boldsymbol{v} = -\boldsymbol{\nabla} \times (\psi \boldsymbol{\nabla} \phi), \tag{A1}$$

which implies $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$. As mentioned earlier, in the case of translational motion and spherical coordinates only the radial component and the polar component of the velocity field are relevant $\boldsymbol{v} = \{v_r, v_\theta, 0\}$. The velocity field in spherical coordinates is expressed in terms of the stream function $\psi(r,\theta)$ as

$$v_r = \frac{-1}{r^2 \sin\theta} \frac{\partial \psi(r,\theta)}{\partial \theta}, \qquad v_\theta = \frac{1}{r \sin\theta} \frac{\partial \psi(r,\theta)}{\partial r}. \tag{A2}$$

The stream function satisfies [39]

$$-\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{v} = \frac{1}{r \sin\theta} E^4 \psi \boldsymbol{e}_\phi \tag{A3}$$

where E^4 is a fourth order partial differential operator

$$E^4 \equiv (E^2)^2, \qquad E^2 = \left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \right) \right]. \tag{A4}$$

In order to account for small viscosity perturbations also the stream function is expressed as $\psi = \psi_0 + \epsilon \psi_1 + \dots$. We take the curl of Eqs. (6a) and (7b). Inserting the stream function leads to scalar equations for the leading and the first order momentum equations

$$O(\epsilon^0): \qquad E^4 \psi_0 = 0 \tag{A5}$$

$$O(\epsilon^1): \qquad E^4 \psi_1 = h_1(r,\theta), \tag{A6}$$

where $h_1(r,\theta) = -r \sin\theta \left(\boldsymbol{\nabla} \eta_1 \times \boldsymbol{\nabla} p_0 + \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \eta_1 \cdot \left[\boldsymbol{\nabla} \boldsymbol{v}_0 + (\boldsymbol{\nabla} \boldsymbol{v}_0)^T \right] \right)_{\boldsymbol{e}_\phi}$ depends on the leading order solution.

Appendix B: Appendix: First order solution for the velocity

First, we express the boundary conditions ((7c) and (7d)) in terms of the stream function ψ_1 . The no-slip boundary conditions require that the first order velocity has to vanish at the surface of a sphere:

$$v_{r1} = \frac{-1}{r^2 \sin\theta} \frac{\partial \psi_1}{\partial \theta} = 0 \ \text{at} \quad |\boldsymbol{r}| = 1 \tag{B1a}$$

$$v_{\theta 1} = \frac{1}{r \sin\theta} \frac{\partial \psi_1}{\partial r} = 0 \ \text{at} \quad |\boldsymbol{r}| = 1 . \tag{B1b}$$

The quiescent fluid in the far-field requires:

$$\frac{\partial_\theta \psi_1}{r^2} \to 0\,, \text{ and } \quad \frac{\partial_r \psi_1}{r} \to 0 \text{ for } \quad |\boldsymbol{r}| \to \infty\,. \tag{B1c}$$

We look for a solution in terms of the Gegenbauer functions $\{\mathscr{I}_n\}_n$ (see C), which are eigenfunctions of the angular part of the E^2 operator, and hence also of the E^4 operator, with the eigenvalues $\{-n(n-1)\}_n$ and provide a complete orthogonal system. Assuming separation of variables we make the ansatz

$$\psi_1(r,\theta) = \sum_{n\geq 2}^{\infty} f_n(r)\mathscr{I}_n(\zeta)\,, \qquad \zeta = \cos\theta. \tag{B2}$$

The corresponding first order velocity field is then obtained from Eq. (A2) and is expressed in terms of Legendre's polynomials P_n and Gegenbauer functions $\mathscr{I}_n$

$$v_{r1} = \frac{1}{r^2}\sum_{n\geq 2} f_n(r)P_n(\zeta) \tag{B3a}$$

$$v_{\theta 1} = \frac{1}{r}\sum_{n\geq 2} f_n'(r)\frac{\mathscr{I}_n(\zeta)}{\sin\theta}\,. \tag{B3b}$$

The restriction $n \geq 2$ refers to the fact that the velocity field has singularities for the modes $n \in \{0,1\}$ at $\theta \in [0,\pi]$ which lead to infinite tangential velocities. Further, we expand the inhomogeneity $h_1(r,\theta)$ in Eq. (A6) in Gegenbauer functions (see Eq. (C5))

$$h_1(r,\theta) = \sum_{n\geq 2} R_n(r)\mathscr{I}_n(\zeta)\,. \tag{B4}$$

The coefficients are calculated using Eq. (C7), from which it follows that the first two modes $R_0(r), R_1(r) = 0$ always vanish. Inserting the ansatz (B3) and the expansion (B4) into Eq. (A6) we obtain

$$\sum_{n\geq 2}\mathscr{I}_n(\zeta)E_n^4(r)f_n(r) = \sum_{l\geq 2} R_l(r)\mathscr{I}_l(\zeta\,),$$

where the differential operator $E_n^4(r)$ is given by

$$E_n^4(r) \equiv \frac{\partial^4}{\partial r^4} - \left(\frac{2}{r^2}\frac{\partial^2}{\partial r^2} - \frac{4}{r^3}\frac{\partial}{\partial r} + \frac{6}{r^4} - \frac{n(n-1)}{r^4}\right)n(n-1). \tag{B5}$$

In order to decouple Gegenbauer modes we apply the orthogonality of the Gegenbauer functions under the scalar product given by Eq. (C2) to obtain

$$\forall_{n\geq 2}: \quad E_n^4(r)f_n(r) = R_n(r)\,. \tag{B6}$$

For given functions $R_n(r)$, which depend on the viscosity variation η_1, each coefficient $f_n(r)$ of the ansatz (B2) can be determined by an ODE with the differential operator $E_n^4(r)$. A solution for an arbitrary inhomogeneity $h_1(r,\theta)$ can be provided by the Green function integration. This requires the knowledge of the Green function for each differential operator $E_n^4(r)$.

Appendix C: Appendix: Gegenbauer functions

The Gegenbauer functions of degree $-1/2$ can be represented by Legendre Polynomials P_n

$$\mathscr{I}_n(\zeta) = \frac{P_{n-2}(\zeta) - P_n(\zeta)}{2n-1} \ . \tag{C1}$$

They are defined on the interval $\zeta \in [-1, 1]$ and for our pupose we choose $\zeta = \cos\theta$ with $\theta \in [0, \pi]$. The Gegenbauer functions are a complete orthogonal system with the relation

$$\begin{aligned} \langle \mathscr{I}_m(\zeta) | \mathscr{I}_n(\zeta) \rangle_\zeta &= \int_{-1}^{1} \frac{\mathscr{I}_n(\zeta)\mathscr{I}_m(\zeta)}{1-\zeta^2} \mathrm{d}\zeta \\ &= \begin{cases} 0\ , & m \neq n \\ \frac{2}{n(n-1)(2n-1)}\ , & m = n \end{cases} \ , \qquad n \geq 2 \end{aligned} \tag{C2}$$

which is not valid for $n \in \{0, 1\}$. Further they satisfy the relation

$$\int_{-1}^{1} \mathscr{I}_n(\zeta)\mathrm{d}\zeta = \begin{cases} 2\ , & n = 0 \\ \frac{2}{3}\ , & n = 2 \\ 0\ , n \neq \{0, 2\} \end{cases} \tag{C3}$$

The first derivative of Gegenbauer functions is

$$\frac{\mathrm{d}\mathscr{I}_n(\zeta)}{\mathrm{d}\zeta} = -P_{n-1}(\zeta) \tag{C4}$$

An axisymmetric function $f(\theta)$ which first through n^{th} derivatives are continuous $f \in C^n([0, \pi])$ can be expressed in terms of Gegenbauer functions and also its $1 - n^{th}$ derivatives

$$f(\theta) = \sum_{l=0}^{\infty} \beta_l\, \mathscr{I}(\zeta) \tag{C5}$$

$$\partial_\theta^k f(\theta) = \sum_{l=0}^{\infty} \beta_l^k\, \mathscr{I}(\zeta)\ , \qquad 1 \leq k \leq n\ . \tag{C6}$$

The coefficients β_l are defined as

$$\begin{aligned} \beta_l &= \frac{1}{2} l(l-1)(2l-1) \langle f(\zeta) | \mathscr{I}_l(\zeta) \rangle_\zeta \\ &= \frac{1}{2} l(l-1)(2l-1) \int_{-1}^{1} f(\zeta) \frac{\mathscr{I}_l(\zeta)}{1-\zeta^2} \mathrm{d}\zeta\ . \end{aligned} \tag{C7}$$

The coefficients β_0, $\beta_1 = 0$ are zero by definition.