Minimum Spectral Bandwidth in Echo Seeded Free Electron Lasers: Supplementary Material

Erik Hemsing
SLAC National Accelerator Laboratory, Menlo Park, California 94025, USA

SUPPLEMENTARY INFORMATION

Second Moment Calculation with Super Gaussian

The integral for a super-Gaussian is,

$$\int z^n e^{-\alpha z^4} dz = \frac{Y_n \Gamma\left(\frac{1}{4}\right)}{2\alpha^{\frac{n+1}{4}}}. \tag{1}$$

where

$$Y_n = \frac{(1+(-1)^n)}{2\Gamma\left(\frac{1}{4}\right)} \Gamma\left(\frac{n+1}{4}\right). \tag{2}$$

Brackets are defined to be shorthand for the normalized expression,

$$\langle z^n \rangle = \frac{\int z^n e^{-\alpha z^4} dz}{\int e^{-\alpha z^4} dz} = \frac{Y_n}{\alpha^{\frac{n}{4}}}. \tag{3}$$

Consider a parameter that can be written as an expansion of the form

$$X(z) = \sum_{N=1}^{\infty} c_N z^N. \tag{4}$$

With these definitions, the second moment of the parameter $X(x)$ is then given by

$$\begin{aligned} \sigma_X^2 &= \langle X^2 \rangle - \langle X \rangle^2 \\ &= \sum_{N,M=1}^{\infty} c_N c_M \left[\langle z^{N+M} \rangle - \langle z^N \rangle \langle z^M \rangle \right] \\ &= \sum_{N,M=1}^{\infty} c_N c_M \frac{Y_{N+M} - Y_N Y_M}{\alpha^{\frac{N+M}{4}}}. \end{aligned} \tag{5}$$

Both (3) and (5) are useful for computing several terms. For instance, the rms length σ_z of the super-Gaussian electron beam is found directly from (3) for $n = 2$, with the direct substitution $\alpha = q^2/\sigma_z^4$ where,

$$q = \frac{\Gamma(3/4)}{\Gamma(1/4)} \approx \frac{1}{3}. \tag{6}$$

Similarly, the transform-limited bunching bandwidth σ_{k_E} can be found by combining expressions and setting $\alpha = 2q^2(1/\sigma_z^4 + 1/2\sigma_g^4)$,

$$\sigma_{k_E}^2 = 3q^2 \sqrt{\frac{2}{\sigma_z^4} + \frac{1}{\sigma_g^4}} \approx \frac{1}{2} \sqrt{\frac{1}{\sigma_z^4} + \frac{1}{2\sigma_g^4}}. \tag{7}$$

The rms width of the bunching envelope is,

$$\sigma_b^2 = \frac{1}{\sqrt{\frac{2}{\sigma_z^4} + \frac{1}{\sigma_g^4}}} = \frac{3q^2}{\sigma_{k_E}^2} \approx \frac{1}{3\sigma_{k_E}^2}. \tag{8}$$

Comparing the series expansion for the phase

$$\varphi(z) = \sum_{N=1}^{\infty} \frac{\phi_N}{N!} z^N, \tag{9}$$

with X above, we see that to calculate $\sigma_{\varphi'}^2$ in the main text we identify $c_N = \phi_N / \Gamma(N)$ to obtain

$$\sigma_{\varphi'}^2 = \sum_{N,M=1}^{\infty} G_{N,M} \frac{\phi_N \phi_M}{(\sigma_{k_E})^{N+M-2}} \tag{10}$$

where

$$G_{N,M} = \frac{Y_{N+M-2} - Y_{N-1} Y_{M-1}}{\Gamma(N)\Gamma(M)} (3q)^{\frac{N+M-2}{2}}. \tag{11}$$

Note that $3q = 1.013\ldots$, so for small N this term can be neglected.

If the phase is described by only a single term in the polynomial expansion, i.e.,

$$\varphi(z) = \frac{\phi_N}{N!} z^N, \tag{12}$$

then the excess bandwidth is given by

$$\sigma_{\varphi'}^2 = \frac{G_N \phi_N^2}{(\sigma_{k_E}^2)^{N-1}} \tag{13}$$

where $G_N = G_{N,N}$,

$$G_N = \frac{Y_{2N-2} - Y_{N-1}^2}{\Gamma(N)^2} (3q)^{N-1}. \tag{14}$$

The first few numerical values are given in the table in the main text.

The total bandwidth is then

$$\sigma_k^2 = \sigma_{k_E}^2 + \frac{G_N \phi_N^2}{(\sigma_{k_E}^2)^{N-1}}. \tag{15}$$

Optimum Laser and Electron Beam for Minimum Bandwidth

The minimum total bandwidth is given when the transform-limited bandwidth satisfies,

$$\left(\sigma_{k_E}^{2N}\right)_{min} = \phi_N^2 G_N (N-1). \tag{16}$$

If the electron beam has a fixed-length and the transform-limited laser pulse length is varied, the optimal laser pulse that minimizes the bandwidth is given by

$$\left(\sigma_L^4\right)_{min} = \frac{m^{4/3}\sigma_z^4}{8\left[(\phi_N\sigma_z^N)^2 G_N(N-1)\right]^{2/N} - 2} \tag{17}$$

Clearly, physical solutions exist only for sufficiently large amplitudes of the nonlinear coefficient,

$$\phi_N^2 G_N(N-1) > \sigma_{k_E}^{2N}(\sigma_L \to \infty) = 1/\sigma_z^{2N} 2^N \tag{18}$$

otherwise, the smallest obtainable bunching bandwidth is the $\sigma_L \to \infty$ limit.

Alternatively, if the laser is held fixed and the electron beam length is adjusted, the optimal length is

$$\left(\sigma_z^4\right)_{min} = \frac{2\sigma_L^4}{8\left[(\phi_N\sigma_L^N)^2 G_N(N-1)\right]^{2/N} - m^{4/3}}. \tag{19}$$

Again, physical solutions only exist if,

$$\phi_N^2 G_N(N-1) > \sigma_{k_E}^{2N}(\sigma_z \to \infty) = m^{2N/3}/\sigma_L^{2N}\sqrt{8^N}. \tag{20}$$

Otherwise, the smallest obtainable bunching bandwidth is the $\sigma_z \to \infty$ limit.