APPENDIX A: PARENT FIRST-ORDER ACTIONS

In this paper, we only consider the familiar "standard" second-order actions of dual graviton and spin-s fields. In this Appendix, we briefly explain the "parent" first-order actions of dual gravity introduced in [44] and spin-s fields in [47].

In general relativity, the Einstein-Hilbert action $S_{\rm EH} = \int d^D x \mathcal{L}_{\rm EH} = \int d^D x R$ can be written in the tetrad formalism as follows

$$\mathcal{L}_{\rm EH}[e] = -e(2\Omega_{ac|b}\Omega^{ab|c} + \Omega_{ab|c}\Omega^{ab|c} - 4\Omega_{ab|}{}^{b}\Omega^{ac|}{}_{c}), \tag{A1}$$

where

$$\Omega_{bc}{}^{a} = e_b{}^{\mu} e_c{}^{\nu}\Omega_{\mu\nu}{}^{a}, \;\; \Omega_{\mu\nu}{}^{a} = \Omega_{[\mu\nu]}{}^{a} = 2\partial_{[\mu}e_{\nu]}{}^{a}. \tag{A2}$$

The tetradic Einstein-Hilbert action (A1) is equivalent to the following parent first-order form in terms of an auxiliary background field $F_{ab|c}$ with mixed symmetry $(2,1)$ that describes Einstein gravity in D dimensions [44]:

$$\mathcal{L}[F,e] = -2e\left(\Omega_{ac|b}F^{ab|c} + \frac{1}{2}F_{ab|c}F^{ab|c} - \frac{1}{2(D-2)}F_{ab|}{}^{b}F^{ac|}{}_{c}\right), \tag{A3}$$

The background field $F_{ab|c}$ can be dualized to a dual field $\bar{F}_{a_1\cdots a_{D-2}|b}$ with mixed symmetry $(D-2,1)$ as follows [44]:

$$F^{ab|c} = \frac{1}{(D-2)!}\epsilon^{abd_1\cdots d_{D-2}}\bar{F}_{d_1\cdots d_{D-2}|}{}^{c}. \tag{A4}$$

This suggests the standard action of gravity in D dimensions [44]:

$$\begin{aligned}\mathcal{L}[\bar{F},e] = -2e\Big(&\frac{1}{(D-2)!}\epsilon^{abd_1\cdots d_{D-2}}\bar{F}_{d_1\cdots d_{D-2}|}{}^{c}\Omega_{ac|b} \\ &+ \frac{1}{2(D-2)!}\bar{F}_{a_1\cdots a_{D-2}|c}\bar{F}^{a_1\cdots a_{D-2}|c} \\ &- \frac{1}{2(D-3)!}\bar{F}_{a_1\cdots a_{D-3}b|}{}^{b}\bar{F}^{a_1\cdots a_{D-3}c|}{}_{c}\Big).\end{aligned} \tag{A5}$$

Considering $e_\mu{}^a = \delta_\mu{}^a + h_\mu{}^a$ around flat space in the linearized theory, Eq. (A2) reduces to $\Omega_{\mu\nu|}{}^{a} = 2\partial_{[\mu}h_{\nu]}{}^{a}$ [134], where $h_\mu{}^a$ corresponds to the graviton $h_a{}^b$ in the $E_D \otimes_s l_1$ non-linear realization (see Section 2.3). The dual field $\bar{F}_{a_1\cdots a_{D-2}|b}$ is also associated with:

$$\bar{F}_{\mu_1\cdots\mu_{D-2}}{}^{a} = (D-2)\partial_{[\mu_1}\tilde{h}_{\mu_2\cdots\mu_{D-2}]}{}^{a}, \tag{A6}$$

where $h_{\mu_1\cdots\mu_{D-3}}{}^{a}$ is a mixed symmetry $(D-3,1)$ tensor corresponds to the dual graviton $h_{a_1\cdots a_{D-3}}{}^{b}$ in D dimensions under the $E_D \otimes_s l_1$ non-linear realization. The parent action (A5) contains the generalized Curtright action (88) that should support the action (87) for the dual graviton in D dimensions [47,134].

Similarly, for spin-s fields we have the following tetradic parent first-order action in D dimensions [47]

$$\begin{aligned}\mathcal{L}[F,e] = -2e\Big(&\Omega_{ac_1|bc_2\cdots c_{s-1}}F^{ab|c_1\cdots c_{s-1}} \\ &+ \frac{(s-1)^2}{s}F_{ab|c_1\cdots c_{s-1}}F^{ab|c_1\cdots c_{s-1}} \\ &+ \frac{(s-1)(s-2)}{s}F_{ab|c_1\cdots c_{s-1}}F^{ac_1|bc_2\cdots c_{s-1}} \\ &+ \frac{(s-1)^2(s-3)}{s(D+s-4)}F_{ab|c_1\cdots c_{s-2}}{}^{b}F^{ad|c_1\cdots c_{s-2}}{}_{d} \\ &+ \frac{(s-1)^2(s-2)}{s(D+s-4)}F_{ab|c_1\cdots c_{s-2}}{}^{a}F^{dc_1|bc_2\cdots c_{s-2}}{}_{d}\Big),\end{aligned} \tag{A7}$$

where $F_{ab|c_1\cdots c_{s-1}}$ is an auxiliary background field, and the field $\Omega_{ab|c_1\cdots c_{s-1}}$ is defined as

$$\Omega_{ab}{}^{c_1\cdots c_{s-1}} = e_a{}^{\mu}e_b{}^{\nu}\Omega_{\mu\nu}{}^{c_1\cdots c_{s-1}}, \tag{A8}$$
$$\Omega_{\mu\nu}{}^{c_1\cdots c_{s-1}} = \Omega_{[\mu\nu]}{}^{c_1\cdots c_{s-1}} = 2\partial_{[\mu}e_{\nu]}{}^{c_1\cdots c_{s-1}}. \tag{A9}$$

In the linearized level, $\Omega_{\mu\nu|}{}^{a_1\cdots a_{s-1}} = 2\partial_{[\mu} h_{\nu]}{}^{a_1\cdots a_{s-1}}$, where $h_\mu{}^{a_1\cdots a_{s-1}}$ describes the spin-s field.

The background field $F_{ab|c_1\cdots c_{s-1}}$ is dualized to a dual field $\bar{F}_{a_1\cdots a_{D-2}|b_1\cdots b_{s-1}}$ as follows:

$$F^{ab|c_1\cdots c_{s-1}} = \frac{1}{(D-2)!}\epsilon^{abd_1\cdots d_{D-2}}\bar{F}_{d_1\cdots d_{D-2}|}{}^{c_1\cdots c_{s-1}}. \tag{A10}$$

This hints at the standard action of spin-s fields in D dimensions [47]:

$$\begin{aligned}\mathcal{L}[\bar{F},e] =& -2e\Big(\frac{1}{(D-2)!}\epsilon^{abd_1\cdots d_{D-2}}\bar{F}_{d_1\cdots d_{D-2}|}{}^{c_1\cdots c_{s-1}}\Omega_{ac_1|bc_2\cdots c_{s-1}} \\ &+ \frac{(s-1)^2}{s(D-2)!}\bar{F}_{a_1\cdots a_{D-2}|c_1\cdots c_{s-1}}\bar{F}^{a_1\cdots a_{D-2}|c_1\cdots c_{s-1}} \\ &+ \frac{(s-1)(s-2)}{s(D-3)!}\bar{F}_{abd_1\cdots d_{D-4}|c_1\cdots c_{s-1}}\bar{F}^{ac_1d_1\cdots d_{D-4}|bc_2\cdots c_{s-1}} \\ &+ \frac{(s-1)^2(s-3)(D-2)}{s(D+s-4)(D-3)!}\bar{F}_{a_1\cdots a_{D-3}b|c_1\cdots c_{s-2}}{}^{b}\bar{F}^{a_1\cdots a_{D-3}d|c_1\cdots c_{s-2}}{}_{d} \\ &+ \frac{(s-1)^2(s-2)(D-2)}{s(D+s-4)(D-3)!}\bar{F}_{abd_1\cdots d_{D-4}|c_1\cdots c_{s-2}}{}^{a}\bar{F}^{dc_1d_1\cdots d_{D-4}|bc_2\cdots c_{s-2}}{}_{d}\Big),\end{aligned} \tag{A11}$$

where $\bar{F}_{a_1\cdots a_{D-2}|b_1\cdots b_{s-1}}$ is associated with the dual spin-s field strength tensor defined as

$$\bar{F}_{\mu_1\cdots\mu_{D-2}}{}^{a_1\cdots a_{s-1}} = (D-2)\partial_{[\mu_1}\tilde{h}_{\mu_2\cdots\mu_{D-2}]}{}^{a_1\cdots a_{s-1}}, \tag{A12}$$

where $\tilde{h}_{\mu_1\cdots\mu_{D-3}}{}^{a_1\cdots a_{s-1}}$ describes the dual spin-s field in D dimensions. We notice that the parent action (A11) includes the action (256), and should support the action (251).