Closed, Two Dimensional Surface Dynamics

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APPENDIX: DERIVATION OF EQUATIONS OF MOTIONS FOR CLOSED, TWO DIMENSIONAL SURFACES

Now we turn to the derivation of (23-25) without using any information from (20-22) (though derivation of (23-25) from (20-22) is strightforwad and trivial if one sets $V^0 = 0$ in (20-22) equations [1]). To deduce the equations of motion we derive the simplest one from the set (23) first. It is direct consequence of generalization of conservation of mass law. Following boundary conditions must be satisfied: 1) at the end of variations $dm/dt = 0$, where $m = \int_S \rho dS$ and 2) a pass integral along any curve γ on the closed surface must vanish $v = n_i V^i = 0$, where n_i is a normal of the curve and lays in the tangent space. Considering these two boundary conditions, Gauss theorem, conservation of mass and integration formula (18), we find:

$$
0 = \int_\gamma v\rho d\gamma = \int_\gamma n_i V^i \rho d\gamma = \int_S \nabla_i(\rho V^i)dS
$$
$$
= \int_S (\nabla_i(\rho V^i) - \rho C B_i^i + \rho C B_i^i)dS
$$
$$
= \int_S (\nabla_i(\rho V^i) - \rho C B_i^i)dS + \int_S \dot{\nabla}\rho dS - \frac{d}{dt}\int_S \rho dS
$$
$$
= \int_S (\dot{\nabla}\rho + \nabla_i(\rho V^i) - \rho C B_i^i)dS \qquad (1)
$$

Since (1) must hold for any integrand the first equation from the set (23) immediately follows. To deduce second and third equations, we take a Lagrangian

$$
L = \int_S \frac{\rho V^2}{2}dS + \int_\Omega (P^+ + \Pi)d\Omega \qquad (2)
$$

and set minimum action principle requesting that $\delta L/\delta t = 0$. Evaluation of space integral is simple and straightforward, using integration theorem for space integral where the convective and advective terms due to the volume motion is properly taken into account (17), we find

$$
\frac{\delta}{\delta t}\int_\Omega (P^+ + \Pi)d\Omega = \int_\Omega \partial_\alpha(P^+ + \Pi)V^\alpha d\Omega + \int_S C(P^+ + \Pi)dS \qquad (3)
$$

Derivation for kinetic part is a bit tricky and challenging that is why we do it last. Straightforward, brute mathematical manipulations, using first equation from (23),

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$$
\frac{\delta}{\delta t}\int_S \frac{\rho V^2}{2}dS = \int_S (\dot{\nabla}\frac{\rho V^2}{2} - C B_i^i \frac{\rho V^2}{2})dS
$$
$$
= \int_S (\dot{\nabla}\rho\frac{V^2}{2} + \rho\dot{\nabla}\frac{V^2}{2} - C B_i^i \frac{\rho V^2}{2})dS
$$
$$
= \int_S ((\rho C B_i^i - \nabla_i(\rho V^i))\frac{V^2}{2} + \rho\dot{\nabla}\frac{V^2}{2} - C B_i^i \frac{\rho V^2}{2})dS
$$
$$
= \int_S (-\nabla_i(\rho V^i)\frac{V^2}{2} + \rho\dot{\nabla}\frac{V^2}{2})dS
$$
$$
= \int_S (-\nabla_i(\rho V^i \frac{V^2}{2}) + \rho V^i \nabla_i \frac{V^2}{2} + \rho\dot{\nabla}\frac{V^2}{2})dS
$$
$$
= \int_S (-\nabla_i(\rho V^i \frac{V^2}{2}) + \rho\vec{V}(V^i\nabla_i\vec{V} + \dot{\nabla}\vec{V}))dS \qquad (4)
$$

At the end of variations when the surface reaches stationary shape according to Gauss theorem (as we used it already in (1)), we find

$$
\int_S -\nabla_i(\rho V^i \frac{V^2}{2})dS = -\int_\gamma \rho V^i n_i \frac{V^2}{2}d\gamma = 0 \qquad (5)
$$

γ is stationary contour of the surface and n_i is the normal to the contour, therefore interface velocity for contour $v = n_i V^i = 0$ and the integral (5) vanishes, correspondingly

$$
\frac{\delta}{\delta t}\int_S \frac{\rho V^2}{2}dS = \int_S \rho\vec{V}(V^i\nabla_i\vec{V} + \dot{\nabla}\vec{V})dS \qquad (6)
$$

To decompose the dot product of the integrand by the normal and the tangential components and, therefore, deduce final equations, we do following algebraic manipulations

$$
\dot{\nabla}\vec{V} + V^i\nabla_i\vec{V}
$$
$$
= \dot{\nabla}\vec{V} + V^i\nabla_i\vec{V} + C V^i B_i^j \vec{S}_j - C V^i B_i^j \vec{S}_j
$$
$$
= \dot{\nabla}\vec{V} + V^i\nabla_i\vec{V} + C V^i B_i^j X_j^\alpha \vec{X}_\alpha - C V^i B_i^j \vec{S}_j \qquad (7)
$$

Now using Weingartens formula $X_j^\alpha B_i^j = -\nabla_i N^\alpha$, metrinilic property of the Euclidian bases $\nabla_i \vec{X}_\alpha = 0$, the definition of the surface normal $\vec{N} = N^\alpha \vec{X}_\alpha$ and taking into account that $\vec{V} = C\vec{N} + V^i \vec{S}_i$ and its derivatives,

we find

$$
\begin{aligned}
\dot{\nabla}\vec{V} &+ V^i\nabla_i\vec{V} + CV^iB_i^jX_j^\alpha\vec{X}_\alpha - CV^iB_i^j\vec{S}_j \\
&= \dot{\nabla}\vec{V} + V^i\nabla_i\vec{V} - CV^i\vec{X}_\alpha\nabla_iN^\alpha - CV^iB_i^j\vec{S}_j \\
&= \dot{\nabla}\vec{V} + V^i\nabla_i\vec{V} - CV^i\nabla_i(N^\alpha\vec{X}_\alpha) - CV^iB_i^j\vec{S}_j \\
&= \dot{\nabla}\vec{V} + V^i\nabla_i\vec{V} - CV^i\nabla_i\vec{N} - CV^iB_i^j\vec{S}_j \\
&= \dot{\nabla}\vec{V} + V^i\nabla_i(C\vec{N}) + V^i\nabla_i(V^j\vec{S}_j) - CV^i\nabla_i\vec{N} \\
&\quad - CV^iB_i^j\vec{S}_j \\
&= \dot{\nabla}\vec{V} + V^i\vec{N}\nabla_iC + V^i\nabla_i(V^j\vec{S}_j) - CV^iB_i^j\vec{S}_j \\
&= \dot{\nabla}(C\vec{N}) + \dot{\nabla}(V^j\vec{S}_j) + V^i\vec{N}\nabla_iC + V^i\nabla_i(V^j\vec{S}_j) \\
&\quad - CV^iB_i^j\vec{S}_j \quad\quad (8)
\end{aligned}
$$

Continuing algebraic manipulations using Thomas formula $\dot{\nabla}\vec{N} = -\nabla^iC\vec{S}_i$, the formula for surface derivative of interface velocity $\vec{N}\nabla_iC = \dot{\nabla}\vec{S}_i$ and the definition of

curvature tensor (5) yield

$$
\begin{aligned}
\dot{\nabla}(C\vec{N}) &+ \dot{\nabla}(V^j\vec{S}_j) + V^i\vec{N}\nabla_iC + V^i\nabla_i(V^j\vec{S}_j) \\
&- CV^iB_i^j\vec{S}_j \\
&= \dot{\nabla}(C\vec{N}) + C\nabla^jC\vec{S}_j + 2V^i\vec{N}\nabla_iC + V^iV^jB_{ij}\vec{N} \\
&\quad + \dot{\nabla}(V^j\vec{S}_j) \\
&\quad - V^i\vec{N}\nabla_iC + V^i\nabla_i(V^j\vec{S}_j) - V^iV^jB_{ij}\vec{N} - C\nabla^jC\vec{S}_j \\
&\quad - CV^iB_i^j\vec{S}_j \\
&= \dot{\nabla}(C\vec{N}) - C\dot{\nabla}\vec{N} + 2V^i\vec{N}\nabla_iC + V^iV^jB_{ij}\vec{N} \\
&\quad + \dot{\nabla}(V^j\vec{S}_j) - V^j\dot{\nabla}\vec{S}_j \\
&\quad + V^i\nabla_i(V^j\vec{S}_j) - V^iV^j\nabla_i\vec{S}_j - C\nabla^jC\vec{S}_j - CV^iB_i^j\vec{S}_j \\
&= (\dot{\nabla}C + 2V^i\nabla_iC + V^iV^jB_{ij})\vec{N} \\
&\quad + (\dot{\nabla}V^j + V^i\nabla_iV^j - C\nabla^jC - CV^iB_i^j)\vec{S}_j \quad (9)
\end{aligned}
$$

Dotting (9) on $\vec{V}$ and taking into account (6) the last derivation finally reveals variation of the kinetic energy, so that we finally find

$$
\frac{\delta}{\delta t}\int_S\frac{\rho V^2}{2}dS = \int_S(\rho C(\dot{\nabla}C + 2V^i\nabla_iC + V^iV^jB_{ij}) + \rho V_i(\dot{\nabla}V^i + V^j\nabla_jV^i - C\nabla^iC - CV^jB_j^i))dS \quad (10)
$$

Combining (1-3) and (10) together and taking into account that the pressure acts on the surface along the surface normal, we immediately find first (23) and the last equation (25) of the set. To clarify second equation (24), we have

$$
\int_S\rho C(\dot{\nabla}C + 2V^i\nabla_iC + V^iV^jB_{ij})dS = \int_\Omega -\partial_\alpha(P^+ + \Pi)V^\alpha d\Omega - \int_S C(P^+ + \Pi)dS
$$
$$
\int_S C(\rho(\dot{\nabla}C + 2V^i\nabla_iC + V^iV^jB_{ij}) + P^+ + \Pi)dS = \int_\Omega -\partial_\alpha(P^+ + \Pi)V^\alpha d\Omega \quad (11)
$$

After applying Gauss theorem to the second equation (11), the surface integral is converted to space integral so that we finally find

$$
\partial_\alpha(\rho V^\alpha(\dot{\nabla}C + 2V^i\nabla_iC + V^iV^jB_{ij}) + (P^+ + \Pi)V^\alpha) = -\partial_\alpha(P^+ + \Pi)V^\alpha \quad (12)
$$

and, therefore, all three equations (23-25) are rigorously clarified.

[1] David V. Svintradze, "Moving manifolds in electromagnetic fields," Frontiers in Physics **5**, 37 (2017).