The mixing of polarizations in the acoustic excitations of disordered media with local isotropy

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1 APPENDIX A

We show that

$$\frac{1}{N!}\frac{d^N}{dz^N}\frac{[z^2]^N}{(z+\tilde{q}_{0i})^{N+1}}|_{z=\tilde{q}_{0i}} = \frac{N+1}{2N+1}\frac{1}{\tilde{q}_{0i}}. \tag{A1}$$

For a generic product of functions $f(z)g(z)$, it is

$$\frac{d^N}{dz^N}[f(z)g(z)] = \sum_{k=0}^{N}\frac{N!}{k!(N-k)!}\frac{d^k}{dz^k}f(q)\frac{d^{N-k}}{dz^{N-k}}g(z). \tag{A2}$$

Furthermore

- $\frac{d^k}{dz^k}z^{2N} = \frac{(2N)!}{(2N-k)!}z^{2N-k}$;
- $\frac{d^{N-k}}{dz^{N-k}}(z+\tilde{q}_{0i})^{-(N+1)} = (-1)^{N-k}\frac{(2N-k)!}{N!}(z+\tilde{q}_{0i})^{-(2N-k+1)}$,

from which we obtain

$$\begin{aligned}
&[\frac{d^k}{dz^k}z^{2N}\frac{d^{N-k}}{dz^{N-k}}(z+\tilde{q}_{0i})^{-(N+1)}]|_{z=\tilde{q}_{0i}} = (-1)^{N-k}\frac{(2N)!}{N!}2^{-(2N-k+1)}\tilde{q}_{0i}^{-1} = (-1)^{N-k}\cdot \\
&\quad\cdot\frac{(2N+1)!(N+1)}{2^{N+1}(N+1)!(2N+1)}2^{-(N-k)}\tilde{q}_{0i}^{-1} = (-1)^{N-k}\frac{(2N)!!(N+1)}{(2N+1)}2^{-(N-k)}\tilde{q}_{0i}^{-1} = \\
&\quad = (-1)^{N-k}\frac{2^N N!(N+1)}{(2N+1)}2^{-(N-k)}\tilde{q}_{0i}^{-1},
\end{aligned} \tag{A3}$$

where the relationships $2^N N! = (2N)!!$ has been exploited, being $n!!$ the double factorial function of the integer n. It follows

$$\frac{1}{N!}\sum_{k=0}^{N}\frac{N!}{k!(N-k)!}[\frac{d^k}{dz^k}z^{2N}\frac{d^{N-k}}{dz^{N-k}}(z+\tilde{q}_{0i})^{-(N+1)}]|_{z=\tilde{q}_{0i}} =$$

$$= \sum_{k=0}^{N}\frac{2^N(N+1)}{(2N+1)}\tilde{q}_{0i}^{-1}\frac{N!}{k!(N-k)!}(-1)^{N-k}2^{(N-k)} = \frac{N+1}{2N+1}\tilde{q}_{0i}^{-1}, \quad \text{(A4)}$$

being from the binomial theorem $\sum_{k=0}^{N}\frac{N!}{k!(N-k)!}(-1)^{N-k}2^{-(N-k)} = \sum_{k=0}^{N}\frac{N!}{k!(N-k)!}(-\frac{1}{2})^{-(N-k)} = (1-\frac{1}{2})^N$.

2 APPENDIX B

Truncation to the second order of the perturbative series expansion can give an approximate expression of the self-energy when the necessary condition $|\mathbf{\Sigma}^3(\mathbf{q},\omega) - \mathbf{\Sigma}^2(\mathbf{q},\omega)| \ll |\mathbf{\Sigma}^2(\mathbf{q},\omega)|$ is satisfied. Because under the hypothesis of local isotropy the self-energy dyadic is diagonal, this inequality is verified if $|\Sigma^3_{kk}(\mathbf{q},\omega) - \Sigma^2_{kk}(\mathbf{q},\omega)| \ll |\Sigma^2_{kk}(\mathbf{q},\omega)|$. In the following we show that the latter inequality holds inside the domain of the (ω, q) plane where the series representation introduced in Sec. 3.1 approximates the quantity $\mathbf{\Sigma}^2(\mathbf{q},\omega)$ if the magnitude of the remainder function of order one of the series representation is small enough. We furthermore show that the necessary condition of validity for the GBA is less stringent than for the Born Approximation.

It is $|\Sigma^3_{kk}(\mathbf{q},\omega) - \Sigma^2_{kk}(\mathbf{q},\omega)| = |\int_{-1}^{1}dxL_{kkii}(x)\frac{2\pi}{c_i^2}\int_0^\infty dq' \;\; q'^2c(q,q',x)[\frac{1}{q_{0i}^2-q'^2-c_i^{-2}\Sigma^2_{ii}(\mathbf{q}',\omega)} - \frac{1}{q_{0i}^2-q'^2-c_i^{-2}\Sigma^1_{ii}(\mathbf{q}',\omega)}]|$. Since, as stated in Theorem I, $\mathrm{Im}[\tilde{\Sigma}^1_{ii}(\mathbf{q},\omega)] > 0$, the function $\frac{1}{q_{0i}^2-q'^2-c_i^{-2}\Sigma^2_{ii}(\mathbf{q}',\omega)}$ can be represented as a Taylor series of argument $\Sigma^2_{ii}(\mathbf{q},\omega) - \Sigma^1_{ii}(\mathbf{q},\omega)$. The integral of the zero-th order term gives $\Sigma^2_{kk}(\mathbf{q},\omega)$. As smaller it is the argument of the Taylor series, as quickly the series converges and as smaller it is the remainder function with respect to the series truncation at a given order. Under the hypotheses of validity of Theorem I and Corollaries I and II and by assuming that the intensity of spatial fluctuations are small enough so that $\tilde{q}_{0i} \approx q_{0i}$, the quantity $\Sigma^2_{ii}(\mathbf{q},\omega) - \Sigma^1_{ii}(\mathbf{q},\omega)$ can be approximated by the remainder function of order one of the series representation for $\Sigma^2_{ii}(\mathbf{q},\omega)$ introduced in Sec. 3.1.1, which we call $S^i_1(\mathbf{q},\omega)$. The condition $|\Sigma^3(\mathbf{q},\omega) - \Sigma^2(\mathbf{q},\omega)| \ll |\Sigma^2(\mathbf{q},\omega)|$ is thus verified when $|S^i_1| \ll 1$.

A necessary condition for the validity of the Born Approximation is $|\Sigma^2_{kk}(\mathbf{q},\omega) - \Sigma^1_{kk}(\mathbf{q},\omega)| \ll |\Sigma^1_{kk}(\mathbf{q},\omega)|$. Under the hypotheses of validity of Theorem I and Corollaries I and II and still by assuming $\tilde{q}_{0i} \approx q_{0i}$ this condition is equivalent to a quick convergence of the series development for $\mathbf{\Sigma}^2(\mathbf{q},\omega)$ introduced in the text, i.e $|\tilde{S}^i_1(\mathbf{q},\omega)| \ll |\tilde{\Sigma}^{n=1}(\mathbf{q},\omega)|$. It is defined $\tilde{S}^i_1(\mathbf{q},\omega) = (\epsilon^2q^2)^{-1}S^i_1(\mathbf{q},\omega)$, with

$$\tilde{S}^i_1(\mathbf{q},\omega) = \int_{-1}^{1}dxL_{iijj}\frac{2\pi}{\tilde{c}_j^2}$$

$$lim_{\eta\to 0^+}\int_0^{q^j_{Max}}dq' \; q'^2 \; c(q,q',x)\frac{1}{(\tilde{q}^2_{0j,\eta}-q'^2)}\cdot\frac{[\frac{\epsilon^2}{\tilde{c}_j^2}q'^2\Delta\tilde{\Sigma}^1_{jj}(\mathbf{q}',\omega_\eta)]^2}{(\tilde{q}^2_{0j,\eta}-q'^2)[\tilde{q}^2_{0j,\eta}-q'^2-\frac{\epsilon^2}{\tilde{c}_j^2}q'^2\Delta\tilde{\Sigma}^1_{jj}(\mathbf{q}',\omega_\eta)]}. \quad \text{(B1)}$$

We observe that

$$|\tilde{S}_1^i(\mathbf{q},\omega)| \leq \int_{-1}^{1} dx L_{iijj} \frac{2\pi}{\tilde{c}_j^2} lim_{\eta\to 0^+} \int_0^{q^j_{Max}} dq'\ q'^2\ c(q,q',x) \frac{1}{|\tilde{q}^2_{0j,\eta} - q'^2|} \cdot \tilde{M}^j \leq$$

$$\leq 2| \int_{-1}^{1} dx L_{iijj} \frac{2\pi}{\tilde{c}_j^2} lim_{\eta\to 0^+} \int_0^{q^j_{Max}} dq'\ q'^2\ c(q,q',x) \frac{1}{(\tilde{q}^2_{0j,\eta} - q'^2)}| \cdot \tilde{M}^j \sim 2|\tilde{\Sigma}^{n=1}_{ii}(\mathbf{q},\omega)|\tilde{M}^j, \quad \text{(B2)}$$

where $\tilde{M}^j = \sup_{q'\in[0,q^j_{Max}]} \left[\frac{[\frac{\epsilon^2}{\tilde{c}_j^2} q'^2 \Delta\tilde{\Sigma}^1_{jj}(\mathbf{q}',\omega_\eta)]^2}{(\tilde{q}^2_{0j,\eta} - q'^2)[\tilde{q}^2_{0j,\eta} - q'^2 - \frac{\epsilon^2}{\tilde{c}_j^2} q'^2 \Delta\tilde{\Sigma}^1_{jj}(\mathbf{q}',\omega_\eta)]}\right]$. The second inequality in Eq. B2 is obtained by similar passages leading to Eqs. 20 and 22 with $N = 0$, being $c(q,q',x) \in \mathbb{R}^+$ and considering that $|\mathrm{Re}[z]| + |\mathrm{Im}[z]| \leq 2|z|$, where z is a complex number. In the case of the Born Approximation it is thus required that $\tilde{M}^j \ll 1$. In the case of the GBA the necessary condition of validity requires $|\tilde{\Sigma}^1_{ii}(\mathbf{q},\omega)|\tilde{M}^j \ll 1$. This condition is less stringent than the former because in the case of small fluctuations when $\Delta\tilde{\Sigma}_{ii}(\mathbf{q},\omega) \sim \tilde{\Sigma}_{ii}(\mathbf{q},\omega)$ it is $|\tilde{\Sigma}^{n=1}_{ii}(\mathbf{q},\omega)| < 1$ under the conditions of validity of Theorem I.