# *Supplementary Material*: The mixing of polarizations in the acoustic excitations of an isotropic random medium

## 1 SUPPLEMENTARY NOTE 1

We show that in the case of an exponential decay of the covariance function it is $|\tilde{R}_k(\mathbf{q},\omega,\epsilon^2)| \lesssim \sum_i \frac{1}{\tilde{c}_i^2}\frac{1}{aq^i_{Max}}$ in the domain of the $(\omega,q)$ plane specified in Sec. 3.2 of the main text , i.e. $q_{0i} \ll q^i_{Max}$ and $q \ll Min_{\{i\}}[q^i_{Max}$.

It can be inferred from Eq. 37 in the main text that for $q \gg q_{0i}$ (i.e. $q \sim q^i_{Max}$),

*a)* $\tilde{q}_{0i}^2 - q^2 < 0$ ;
*b)* $Re\{\Delta\tilde{\Sigma}^1_i(\mathbf{q},\omega)\} < 0$;
*c)* $|Re\{\Delta\tilde{\Sigma}^1_i(\mathbf{q},\omega)\}|$ definitively increases by increasing $q$ with a $q^2$ leading term (see Fig. 2);
*d)* $Im\{\Delta\tilde{\Sigma}^1_i(\mathbf{q},\omega)\} > 0$;
*e)* $Im\{\Delta\tilde{\Sigma}^1_i(\mathbf{q},\omega)\} \ll 1$.

Point *a)* is valid because $q_{0i} \sim \tilde{q}_{0i}$ when $\epsilon^2$ is small. **To** support of point *e)*, Fig. S1 shows the wavevector trend of $|Re\{\Delta\tilde{\Sigma}^1_i(\mathbf{q},\omega)\}|$, $|Im\{\Delta\tilde{\Sigma}^1_i(\mathbf{q},\omega)\}|$ and $|\Delta\tilde{\Sigma}^1_i(\mathbf{q},\omega)|$ for a given value of $q_{0i} \ll q^i_{Max}$. For $q \sim q^i_{Max}$ we observe that the value of $|Im\{\Delta\tilde{\Sigma}^1_i(\mathbf{q},\omega)\}|$ is negligible with respect to the value of $|Re\{\Delta\tilde{\Sigma}^1_i(\mathbf{q},\omega)\}|$. We will call the wavevectors region : $q \sim q^i_{Max}$, the 'high-$q$' region.

In the 'high-$q$' region,

i) the function $\frac{1}{|\tilde{q}_{0i}^2 - q^2 - \frac{\epsilon^2}{\tilde{c}_i^2}q^2\Delta\tilde{\Sigma}^1_i(\mathbf{q},\omega)|}$ has a maximum in $\overline{q}_i$ : $\frac{\epsilon^2}{\tilde{c}_i^2}|Re\{\Delta\tilde{\Sigma}^1_i(\overline{\mathbf{q}}_i,\omega)\}| = 1 - \frac{\tilde{q}_{0i}^2}{\overline{q}_i^2}$;
ii) the function $\frac{1}{|\tilde{q}_{0i}^2 - q^2 - \frac{\epsilon^2}{\tilde{c}_i^2}q^2\Delta\tilde{\Sigma}^1_i(\mathbf{q},\omega)|}$ monotonically decreases by increasing $q$ for $q > \overline{q}_i$.
iii) it is $\overline{q}_i \le q^i_{Max}$;
iv) for $q > \overline{q}^i_{Max}$, where $\overline{q}^i_{Max}$ : $\frac{\epsilon^2}{\tilde{c}_i^2}|Re\{\Delta\tilde{\Sigma}^1_i(\overline{\mathbf{q}}^i_{Max},\omega)\}| = 2[1 - \frac{\tilde{q}_{0i}^2}{(\overline{q}^i_{Max})^2}]$, it is $\frac{1}{|\tilde{q}_{0i}^2 - q^2 - \frac{\epsilon^2}{\tilde{c}_i^2}q^2\Delta\tilde{\Sigma}^1_i(\mathbf{q},\omega)|} \le \frac{1}{|\tilde{q}_{0i}^2 - q^2|}$.

The behavior described in points *i)* and *ii)* can be furthrmore observed in Figure 1, Panels 3, in the main text. We prove in the following point *iii)*. The other points follow immediately from points *a)-e)*. Since $\overline{q}_i$ belongs to the 'high-$q$' region, it is $\tilde{q}_{0i} < \overline{q}_i$ and, furthermore, it follows from point *e)* that $\frac{\epsilon^2}{\tilde{c}_i^2}|\Delta\tilde{\Sigma}^1_i(\overline{\mathbf{q}}_i,\omega)| \sim \frac{\epsilon^2}{\tilde{c}_i^2}|Re\{\Delta\tilde{\Sigma}^1_i(\overline{\mathbf{q}}_i,\omega)\}| = 1 - \frac{\tilde{q}_{0i}^2}{\overline{q}_i^2} \le 1$ (equal to 1 in the limit $\frac{\tilde{q}_{0i}}{\overline{q}_i} \to 0$). It is thus $\frac{\epsilon^2}{\tilde{c}_i^2}|\Delta\tilde{\Sigma}^1_i(\overline{\mathbf{q}}_i,\omega)| \le 1$, from which it follows $\overline{q}_i \le q^i_{Max}$.

In the following we define an upper bound for $|\tilde{R}_k(\mathbf{q},\omega,\epsilon^2)|$. First, we estimate the contribution to the rest function for $q > q^{i*}_{Max} = max[q^i_{Max}, \overline{q}^i_{Max}]$. We will call it $\tilde{R}^*_k(\mathbf{q},\omega,\epsilon^2) =$

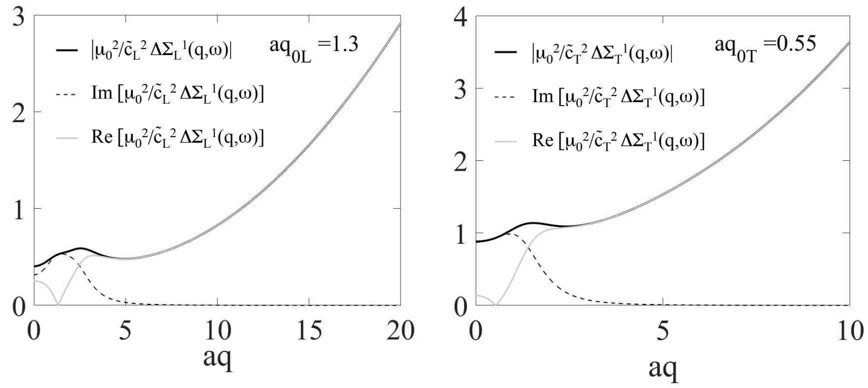


**Figure S1.** $\frac{\mu_0^2}{\tilde{c}^2_{L(T)}}|\Delta\tilde{\Sigma}^1_{L(T)}(\mathbf{q},\omega)|$ (black line), $\mathrm{Im}[\frac{\mu_0^2}{\tilde{c}^2_{L(T)}}\Delta\tilde{\Sigma}^1_{L(T)}(\mathbf{q},\omega)]$ (dashed line) and $|\mathrm{Re}[\frac{\mu_0^2}{\tilde{c}^2_{L(T)}}\Delta\tilde{\Sigma}^1_{L(T)}(\mathbf{q},\omega)]|$ (grey line) as a function of $q$ for a fixed value of $aq_{0L(T)} \ll aq^{L(T)}_{Max}$. The covariance function is an exponential decay function. The theory's input parameters are listed in the main text.

$\int_{-1}^{1} dx L_{kkii}(x)\frac{2\pi}{\tilde{c}_i^2}\tilde{r}_i^*(\mathbf{q},\omega,\epsilon^2,x)$. From points $a)$ and $iv)$ it follows that

$$|\tilde{r}_i^*(\mathbf{q},\omega,\epsilon^2,x)| = |\int_{q^{i*}_{Max}}^{\infty} dq' c(q,q',x)\frac{{q'}^2}{\tilde{q}_{0i}^2 - q'^2 - \frac{\epsilon^2}{\tilde{c}_i^2}q'^2\Delta\tilde{\Sigma}^1_i(\mathbf{q}',\omega)}| \leq \int_{q^{i*}_{Max}}^{\infty} dq' c(q,q',x)\frac{{q'}^2}{q'^2-\tilde{q}_{0i}^2}. \quad \text{(S1)}$$

It is $\int_{q^{i*}_{Max}}^{\infty} dq' c(q,q',x)\frac{{q'}^2}{q'^2-\tilde{q}^2_{0i}} \sim \frac{1}{\pi^2}\frac{1}{aq^{i*}_{Max}} \leq \frac{1}{aq^i_{Max}}$, by recalling that $q^{i*}_{Max} \sim q^i_{Max}$, $c(q,q',x) = \frac{a}{\pi^2}\frac{(aq')^2}{(1+(aq')^2+(aq)^2-2(aq)(aq')x)^2}$, $aq^i_{Max} \gg 1$ for small values of $\frac{\epsilon^2}{\tilde{c}_i^2}$, and $q, q_{0i} \ll q^i_{Max}$. Because $\int_{-1}^{1} dx|L_{kkii}(x)| = O(1)$, finally it is $|\tilde{R}^*_k(\mathbf{q},\omega,\epsilon^2)| \lesssim \sum_i \frac{2}{\pi}\frac{1}{\tilde{c}_i^2}\frac{1}{aq^i_{Max}}$.

If $q^{i*}_{Max} = \overline{q}^i_{Max} > q^i_{Max}$, we need, in addition, to estimate the order of magnitude of

$$|\int_{q^i_{Max}}^{\overline{q}^i_{Max}} dq' c(q,q',x)\frac{{q'}^2}{\tilde{q}^2_{0i}-q'^2-\frac{\epsilon^2}{\tilde{c}_i^2}q'^2\Delta\tilde{\Sigma}^1_i(\mathbf{q}',\omega)}|. \quad \text{(S2)}$$

We instead take into account the following integral

$$\tilde{r}^\delta_i(x,\mathbf{q},\omega,\epsilon^2) = \int_{\overline{q}_i-\delta^i}^{\overline{q}_i+\delta^i} dq' c(q,q',x)\frac{{q'}^2}{\tilde{q}^2_{0i}-q'^2-\frac{\epsilon^2}{\tilde{c}_i^2}q'^2\Delta\tilde{\Sigma}^1_i(\mathbf{q}',\omega)}, \quad \text{(S3)}$$

where $\delta^i = \overline{q}^i_{Max} - \overline{q}_i$. It is $[q^i_{Max}, \overline{q}^i_{Max}] \subset [\overline{q}_i - \delta^i, \overline{q}_i + \delta^i]$ because $\overline{q}_i \leq q^i_{Max}$. We show in the following that $O(|\tilde{r}^\delta_i(x,\mathbf{q},\omega,\epsilon^2)|) = O(|\tilde{r}^*_i(x,\mathbf{q},\omega,\epsilon^2)|)$. Since the power series expansion of $< G(\mathbf{q},\omega) >^1$ defined in Theorem I in the main text converges a.e. for $q < \overline{q}_i - \delta^i < q^i_{Max}$, we can refix the upper integration boundary of the integral in Corrolary II (Eq. 34 in the main text) to $\overline{q}_i - \delta^i$. This will not affect the domain of the $(\omega, q)$ plane where the GBA can be applied because the contribution to the integral defining the self-energy for $q' \in [\overline{q}_i - \delta^i, \overline{q}_i + \delta^i = \overline{q}^i_{Max}]$ results to be of the same order of magnitude of the remainder function. The function $\frac{1}{|\tilde{q}^2_{0i}-q^2-\frac{\epsilon^2}{\tilde{c}_i^2}q^2\Delta\tilde{\Sigma}^1_i(\mathbf{q},\omega)|}$ in the 'high-$q$' region has a local maximum in

$\overline{q}_i$, as described in point *i)*, see also Figure 1, Panels 3, in the main text. This function is a peak-like function centered in $\overline{q}_i$. We notice that $\tilde{q}_{0i}^2 - \overline{q}_i^2 - \frac{\epsilon^2}{\tilde{c}_i^2}\overline{q}_i^2 Re\{\Delta\tilde{\Sigma}_i^1(\overline{q}_i,\omega)\} = 0$. In a neighbor of $\overline{q}_i$ small enough we can thus make the following approximation $1 - \frac{\epsilon^2}{\tilde{c}_i^2}\frac{q^2}{\tilde{q}_{0i}^2-q^2}Re\{\Delta\tilde{\Sigma}_i^1(\mathbf{q},\omega)\} \sim A_i(q-\overline{q}_i)$, where $A_i = \frac{d}{dq}[1 - \frac{\epsilon^2}{\tilde{c}_i^2}\frac{q^2}{\tilde{q}_{0i}^2-q^2}Re\{\Delta\tilde{\Sigma}_i^1(\mathbf{q},\omega)\}]|_{q=\overline{q}_i}$. An approximate expression for the constant $A_i$ can be obtained by considering that in the 'high-$q$' region $Re\{\Delta\tilde{\Sigma}_i^1(\mathbf{q},\omega)\} \sim -C_i q^2$, as it can be derived from points *b)* and *c)*. The constant $C_i$ should satisfy the condition

$$1 - \frac{\epsilon^2}{\tilde{c}_i^2}\frac{\overline{q}_i^2}{\tilde{q}_{0i}^2-\overline{q}_i^2}Re\{\Delta\tilde{\Sigma}_i^1(\overline{q}_i,\omega)\} \sim 1 - \frac{\epsilon^2}{\tilde{c}_i^2}C_i\overline{q}_i^2 = 0,$$

being $\tilde{q}_{0i} \ll \overline{q}_i$. It is thus $A_i \sim -\frac{\epsilon^2}{\tilde{c}_i^2}C_i 2\overline{q}_i = -\frac{2}{\overline{q}_i}$. We define $\eta_i = -\frac{\epsilon^2}{\tilde{c}_i^2}\frac{\overline{q}_i^2}{\tilde{q}_{0i}^2-\overline{q}_i^2}Im\{\Delta\tilde{\Sigma}_i^1(\overline{q}_i,\omega)\} \sim \frac{\epsilon^2}{\tilde{c}_i^2}Im\{\Delta\tilde{\Sigma}_i^1(\overline{q}_i,\omega)\} > 0$, see point *d)*. Furthermore from point *e)* it follows that $\eta_i \ll 1$. In the interval $[\overline{q}_i - \delta^i, \overline{q}_i + \delta^i]$ we can thus take

$$\frac{1}{\tilde{q}_{0i}^2 - q^2 - \frac{\epsilon^2}{\tilde{c}_i^2}q^2\Delta\tilde{\Sigma}_i^1(\mathbf{q},\omega)} \sim \frac{1}{\tilde{q}_{0i}^2-q^2}\frac{1}{A_i(q-\overline{q}_i)+i\eta_i}. \tag{S4}$$

It follows that

$$\begin{aligned}|\tilde{r}_i^\delta(\mathbf{q},\omega,\epsilon^2,x)| \sim \Big| \int_{\overline{q}_i-\delta^i}^{\overline{q}_i+\delta^i} dq' c(q,q',x)\frac{{q'}^2}{\tilde{q}_{0i}^2 - q'^2}\frac{A_i(q'-\overline{q}_i)}{[A_i(q'-\overline{q}_i)]^2+\eta_i^2} - \\ - i\int_{\overline{q}_i-\delta^i}^{\overline{q}_i+\delta^i} dq' c(q,q',x)\frac{{q'}^2}{\tilde{q}_{0i}^2 - q'^2}\frac{\eta_i}{[A_i(q'-\overline{q}_i)]^2+\eta_i^2}\Big| \sim \frac{\pi}{|A_i|}c(q,\overline{q}_i,x)\frac{{\overline{q}_i}^2}{\overline{q}_i^2 - \tilde{q}_{0i}^2} \sim \frac{1}{2\pi\ a\overline{q}_i}.\end{aligned} \tag{S5}$$

We assumed that in the integration interval $c(q,q',x)\frac{{q'}^2}{q'^2-\tilde{q}_{0i}^2} \sim c(q,\overline{q}_i,x)\frac{{\overline{q}_i}^2}{\overline{q}_i^2-\tilde{q}_{0i}^2} \sim \frac{a}{\pi^2}\frac{1}{(a\overline{q}_i)^2}$ since $\overline{q}_i$ belongs to the 'high-$q$' region and consequently $a\overline{q}_i \gg 1$, $q \ll \overline{q}_i$, $q_{0i} \ll \overline{q}_i$. Furthermore, we observe that the integrand of the first integral in Eq. S5 is symmetric with respect to the center of the integration interval. This integral is thus zero. The integrand of the second integral is a Lorentz function of area $\frac{\pi}{|A_i|}$. We finally considered that $\eta_i \ll \delta^i$, as follows from point *e)*. Because both $\overline{q}_i$ and $q_{Max}^i$ belong to the 'high-$q$' region we can finally assume $\frac{1}{a\overline{q}_i} \sim \frac{1}{aq_{Max}^i}$.

## 2 SUPPLEMENTARY NOTE 2

We provide a numerical estimation of the absolute value of the remainder function related to the GBA for given values of frequency and wavevector. We furthermore verify by a numerical estimation that the absolute value of the term $F_k^1(\mathbf{q},\omega)$ is significantly larger than such a value. Finally we numerically verify the consistency of the approximation $\Delta\Sigma^1(\mathbf{q},\omega) \approx \Delta\Sigma^1(0,\omega)$ while calculating $F_k^1(\mathbf{q},\omega)$. To this aim we numerically computed the following integrals, with $aq_{0L} = 1.3$ and $aq = 1.2$,

$$\tilde{\Sigma}_{LL}(\mathbf{q},\omega)_{<q_{Max}^L} = \int_{-1}^1 dx L_{LL}(x)\frac{2\pi}{\tilde{c}_L^2}\int_0^{q_{Max}^L} dq'\ {q'}^2 c(q,q',x)\frac{1}{\tilde{q}_{0L}^2 - q'^2 - \frac{\epsilon^2}{\tilde{c}_L^2}q'^2\Delta\tilde{\Sigma}_L^1(\mathbf{q}',\omega)};$$

$$\tilde{R}_{LL}(\mathbf{q},\omega,\epsilon^2) = \int_{-1}^1 dx L_{LL}(x)\frac{2\pi}{\tilde{c}_L^2}\int_{q_{Max}^L}^\infty dq'\ {q'}^2 c(q,q',x)\frac{1}{\tilde{q}_{0L}^2 - q'^2 - \frac{\epsilon^2}{\tilde{c}_L^2}q'^2\Delta\tilde{\Sigma}_L^1(\mathbf{q}',\omega)}.$$

The theory's input parameters are the same listed in the main text. For such input parameters it is $aq^L_{Max} = 18$, as it is possible to observe in Fig. 1, Panel 2 -a) in the main text. We obtain $\frac{\mu_0^2}{{c_L^0}^2}|\tilde{\Sigma}_{LL\,<q^L_{Max}}(aq_{0L} = 1.3, aq = 1.2)| = 3.3 \cdot 10^{-1}$ and $\frac{\mu_0^2}{{c_L^0}^2}|\tilde{R}_{LL}(aq_{0L} = 1.3, aq = 1.2)| = 1.0 \cdot 10^{-3}$. We can compare the latter quantity with the upper bound estimation for $|\tilde{R}_{LL}|$, given in the main text and assessed in Supplemenatry Note 1, i.e. $\sim \frac{\mu_0^2}{{c_L^0}^2}\frac{1}{\tilde{c}_L^2}\frac{2}{\pi}\frac{1}{aq^L_{Max}} = 3.0 \cdot 10^{-3}$. We furthermore numerically evaluate the quantity

$$F^1_{LL}(\mathbf{q},\omega) = \int_{-1}^{1} dx L_{LL}(x)\frac{2\pi}{\tilde{c}_L^2}\int_0^{q^L_{Max}} dq'\, {q'}^2 c(q,q',x)\frac{\frac{\epsilon^2}{\tilde{c}_L^2}{q'}^2\Delta\tilde{\Sigma}_L(\mathbf{q}',\omega)}{(\tilde{q}_{0L}^2 - q'^2)^2},$$

achieving [1] $\frac{\mu_0^2}{{c_L^0}^2}|F^1_{LL}(aq_{0L} = 1.3, aq = 1.2)| = 6.6 \cdot 10^{-2}$, which is a value significantly larger than $|\tilde{R}_{LL}(aq_{0L} = 1.3, aq = 1.2)|$.

We finally numerically calculate $\frac{\mu_0^2}{{c_L^0}^2}|F^{1*}_{LL}(aq_{0L} = 1.3, aq = 1.2)| = 9.0 \cdot 10^{-2}$, where $F^{1*}_{LL}$ is equal to $F^1_{LL}$ but $\Delta\tilde{\Sigma}^1_L(\mathbf{q},\omega)$ is replaced by $\Delta\tilde{\Sigma}^1_L(0,\omega)$. This latter can be compared with the numerical evaluation of $\frac{\mu_0^2}{{c_L^0}^2}|F^1_{LL}(aq_{0L} = 1.3, aq = 1.2)|$ obtained above, i.e. $6.6 \cdot 10^{-2}$.

## 3 SUPPLEMENTARY NOTE 3

We show that when in the domain of validity of the GBA the approximation $\Delta\tilde{\Sigma}^1_{ii}(\mathbf{q}',\omega) \sim \Delta\tilde{\Sigma}^1_{ii}(0,\omega)$ holds, it is possible to extend the upper integration boundary of the integral defining $F^1_k(\mathbf{q},\omega)$ to infinity. The related error is of the same order of magnitude of $|\tilde{R}_k(\mathbf{q},\omega,\epsilon^2)|$. We observe that

$$\int_{q^i_{Max}}^{\infty} dq'\, {q'}^2 c(q,q',x)\frac{q'^2}{(q'^2-\tilde{q}_{0i}^2)^2}|\frac{\epsilon^2}{\tilde{c}_i^2}\Delta\tilde{\Sigma}^1_{ii}(0,\omega)| \;\leq \int_{q^i_{Max}}^{\infty} dq'\, {q'}^2 c(q,q',x)\frac{q'^2}{(q'^2-\tilde{q}_{0i}^2)^2} \sim \frac{1}{aq^i_{Max}}. \tag{S6}$$

In the region of frequency where $\frac{\epsilon^2}{\tilde{c}_i^2}\Delta\tilde{\Sigma}^{1,Max}_i(\omega) < 1$ indeed it is $\frac{\epsilon^2}{\tilde{c}_i^2}|\Delta\tilde{\Sigma}_{ii}(0,\omega)| < 1$. Furthermore for $q' > q^i_{Max}$, it is $\tilde{q}_{0i} \ll q'$. If the approximation $\Delta\tilde{\Sigma}^1_{ii}(\mathbf{q}',\omega) \sim \Delta\tilde{\Sigma}^1_{ii}(0,\omega)$ holds, the Hadamard Principal value of the integral defining $F^1_k(\mathbf{q},\omega)$ can be calculated by exploiting the Residue Theorem because the function $z^2 c(q,z,x)\frac{[z^2\frac{\epsilon^2}{\tilde{c}_i^2}\Delta\tilde{\Sigma}^1_{ii}(0,\omega)]}{(z^2-\tilde{q}_{0i}^2)^2}$ has only non-essential singularities in the complex plane.

[1] We take the definition $\#\int_a^b \frac{f(x)}{(x-x_0)^2} = lim_{\eta\to 0}\big[\int_a^{x_0-\eta}\frac{f(x)}{(x-x_0)^2}dx + \int_{x_0+\eta}^b\frac{f(x)}{(x-x_0)^2}dx - \frac{2f(x_0)}{\eta}\big]$. It is $lim_{\eta\to 0}\big[\int_{\tilde{q}_{0i}-\eta}^{\tilde{q}_{0i}+\eta} q'^2 c(q,q',x)\frac{q'^2\Delta\tilde{\Sigma}_i(\mathbf{q}',\omega)}{(q'^2-\tilde{q}_{0i}^2)^2}dq' - \tilde{q}_{0i}^2 c(q,\tilde{q}_0,x)\frac{\Delta\tilde{\Sigma}_i(\tilde{q}_{0i},\omega)}{2\eta}\big] = 0$. Indeed, given the continuity of the function $c(q,q',x)$ and $\Delta\tilde{\Sigma}_i(\mathbf{q}',\omega)$ in $\tilde{q}_{0i}$, the quantity in the brackets can be approximated by the expression $\xi(\eta) = c(q,\tilde{q}_0,x)\tilde{q}_{0i}^2\Delta\tilde{\Sigma}_i(\tilde{q}_{0i},\omega)\big[\int_{\tilde{q}_{0i}-\eta}^{\tilde{q}_{0i}+\eta}\frac{q'^2}{(\tilde{q}_{0i}^2-q'^2)^2}dq' - \frac{1}{2\eta}\big] = lim_{\eta\to 0}\big[\frac{1}{2\eta}(1-\frac{4\tilde{q}_{0i}^2-2\eta^2}{4\tilde{q}_{0i}^2-\eta^2}) - ln(\frac{2\tilde{q}_{0i}-\eta}{2\tilde{q}_{0i}+\eta})\big]$. It is hence $lim_{\eta\to 0}\xi(\eta) = 0$. In the numerical computation we assume $\#\int_0^{q_{Max}} q'^2 c(q,q',x)\frac{\frac{\epsilon^2}{\tilde{c}_i^2}q'^2\Delta\tilde{\Sigma}_i(\mathbf{q}',\omega)}{(\tilde{q}_{0i}^2-q'^2)^2}dq' = \int_0^{\tilde{q}_{0i}-0.5} q'^2 c(q,q',x)\frac{\frac{\epsilon^2}{\tilde{c}_i^2}q'^2\Delta\tilde{\Sigma}_i(\mathbf{q}',\omega)}{(\tilde{q}_{0i}^2-q'^2)^2}dq' + \int_{\tilde{q}_{0i}-0.5}^{q_{Max}} q'^2 c(q,q',x)\frac{\frac{\epsilon^2}{\tilde{c}^2}q^2\Delta\tilde{\Sigma}_i(\mathbf{q}',\omega)}{(\tilde{q}_{0i}^2-q'^2)^2}dq'$. We can take the quantity $\xi(\eta = 0.5) = 0.005$ as the related error.

# 4 SUPPLEMENTARY NOTE 4

We show that as long as the condition $|\frac{\Delta\tilde{\Sigma}^1_{ii}(\mathbf{q}',\omega)-\Delta\tilde{\Sigma}^1_{ii}(0,\omega)}{\Delta\tilde{\Sigma}^1_{ii}(0,\omega)}| < \frac{1}{2}$ is fulfilled the dominant contribution to the integral defing $F^1_k(\mathbf{q},\omega)$ can be obtained trough the approximation $\Delta\tilde{\Sigma}^1_{ii}(\mathbf{q}',\omega) \sim \Delta\tilde{\Sigma}^1_{ii}(0,\omega)$. It is

$$F^1_k(\mathbf{q},\omega) = lim_{\eta\to 0^+}\int_{-1}^{1} dx L_{kkii}(x)\frac{2\pi}{\tilde{c}_i^2}\int_0^{q^i_{Max}} dq'\, {q'}^2 c(q,q',x)\cdot$$
$$\cdot\{\frac{\frac{\epsilon^2}{\tilde{c}_i^2}q'^2\Delta\tilde{\Sigma}_{ii}(0,\omega_\eta)}{(\tilde{q}^2_{0i,\eta}-q'^2)^2} + \frac{\frac{\epsilon^2}{\tilde{c}^2}q'^2[\Delta\tilde{\Sigma}_{ii}(\mathbf{q}',\omega_\eta)-\Delta\tilde{\Sigma}_{ii}(0,\omega_\eta)]}{(\tilde{q}^2_{0i,\eta}-q'^2)^2}\}. \tag{S7}$$

If $|\frac{\Delta\tilde{\Sigma}^1_{ii}(\mathbf{q},\omega)-\Delta\tilde{\Sigma}^1_{ii}(0,\omega)}{\Delta\tilde{\Sigma}^1_{ii}(0,\omega)}| < \frac{1}{2}$ in the integration interval $[0, q^i_{Max}]$ the integral of the first term of the summation in Eq. S7 is the dominant. In support of this statement we observe that $lim_{\eta\to 0^+}\int_0^{q^i_{Max}} dq' \quad q'^2 c(q,q',x)\frac{1}{2}\frac{\frac{\epsilon^2}{\tilde{c}^2}q'^2}{|\tilde{q}^2_{0i,\eta}-q'^2|^2} \leq lim_{\eta\to 0^+}|\int_0^{q^i_{Max}} dq' \quad q'^2 c(q,q',x)\frac{\frac{\epsilon^2}{\tilde{c}^2}q'^2}{(\tilde{q}^2_{0i,\eta}-q'^2)^2}|$. The inequality is obtained by exploiting Eqs. 22 and 24 with $N=1$ in the text, recalling that $c(q,q',x)\in\mathbb{R}^+$ and $|\mathrm{Re}[z]| + |\mathrm{Im}[z]| \leq 2|z|$, where z is a complex number.